Discover Calculus

Single-Variable Calculus Topics with Motivating Activities

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Contents

Chapter 1

Limits

1.1 Introduction to Limits

Almost 2,500 years ago, the Greek philosopher Zeno of Elea gifted the world with a set of philosophical paradoxes that provide the foundation for how we will begin our study of calculus. Perhaps the most famous of Zeno's paradoxes is the paradox of Achilles and the Tortoise.

1.1.1 Achilles and the Tortoise

In the paradox of Achilles and the Tortoise, the Greek hero Achilles is in a race with a tortoise. Obviously the tortoise is much slower than Achilles, and so the tortoise gets a 100m head start. The race begins, and while the tortoise moves as quickly as it can, Achilles has the obvious advantage. They both are running at a constant speed, with Achilles running faster. Achilles runs 100m, catching up to the tortoise's starting point.

In the meantime, the tortoise has moved 2 meters. Achilles has almost caught up and passed the tortoise at this point! In a *very* short time, Achilles is able to run the 2 meters to catch up to where the tortoise was. Unfortunately, in that short amount of time, the tortoise has kept on moving, and is farther along by now.

Every time Achilles catches up to where the tortoise was, the tortoise has moved farther along, and Achilles has to keep catching up.

Can Achilles, the paragon of athleticism, ever catch the tortoise?

1.1.2 A Modern Retelling

A college student is excited about having enrolled in their first calculus class. On the first day of class, they head to class. When they enter the hallway with their classroom at the end, they take a breath and excitedly head to class.

In order to get to class, though, they have to travel halfway down the hallway. Almost there.

Now, to go the rest of the way, the student will half to get to the point that is halfway between them and the door. Getting closer.

They're getting excited. Finally, their first calculus class! But to get to the class, they have to reach the point halfway between them and the door.

Their smile starts fading. They repeat the process, and go halfway from their position to the door. They're closer, but not there yet.

If they keep having to reach the new halfway point, and that halfway point is never actually *at* the door, then will they ever get there?

Halfway to the door, then halfway again, closer and closer without ever getting there.

Will the student ever get there, or are they doomed to keep getting closer and closer without ever reaching the door?

1.2 The Definition of the Limit

1.2.1 Defining a Limit

Activity 1.2.1 Close or Not? We're going to try to think how we might define "close"-ness as a property, but, more importantly, we're going to try to realize the struggle of creating definitions in a mathematical context. We want our definition to be meaningful, precise, and useful, and those are hard goals to reach! Coming to some agreement on this is a particularly tricky task.

(a) For each of the following pairs of things, decide on which pairs you would classify as "close" to each other.

- You, right now, and the nearest city with a population of 1 million or higher
- Your two nostrils
- You and the door of the room you are in
- You and the person nearest you
- The floor of the room you are in and the ceiling of the room you are in

(b) For your classification of "close," what does "close" mean? Finish the sentence: A pair of objects are *close* to each other if...

(c) Let's think about how close two things would have to be in order to satisfy everyone's definition of "close." Pick two objects that you think everyone would agree are "close," if by "everyone" we meant:

- All of the people in the building you are in right now.
- All of the people in the city that you are in right now.
- All of the people in the country that you are in right now.
- Everyone, everywhere, all at once.

(d) Let's put ourselves into the context of functions and numbers. Consider the linear function $y = 4x - 1$. Our goal is to find some x-values that, when we put them into our function, give us y-value outputs that are "close" to the number 2. You get to define what close means.

First, evaluate $f(0)$ and $f(1)$. Are these y-values "close" to 2, in your definition of "close?"

(e) Pick five more, different, numbers that are "close" to 2 in your definition of "close." For each one, find the x-values that give you those y-values.

(f) How far away from $x = \frac{3}{4}$ can you go and still have y-value outputs that are "close" to 2?

To wrap this up, think about your points that you have: you have a list of x-coordinates that are clustered around $x = \frac{3}{4}$ where, when you evaluate $y = 4x - 1$ at those x-values, you get y-values that are "close" to 2. Great!

Do you think others will agree? Or do you think that other people might look at your list of y-values and decide that some of them *aren't* close to 2?

Do you think you would agree with other peoples' lists? Or you do think that you might look at other peoples' lists of y-values and decide that some of them *aren't* close to 2?

Definition 1.2.1 Limit of a Function. For the function $f(x)$ defined at all x-xalues around a (except maybe at $x = a$ itself), we say that the **limit of** $f(x)$ as x approaches a is L if $f(x)$ is arbitrarily close to the single, real number L whenever x is sufficiently close to, but not equal to, a. We write this as:

$$\lim_{x \to a} f(x) = L$$

or sometimes we write $f(x) \to L$ when $x \to a$. ◊

When we say "around $x = a$", we really just mean on either side of it. We can clarify if we want.

Definition 1.2.2 Left-Sided Limit. For the function $f(x)$ defined at all x-xalues around and less than a, we say that the **left-sided limit of** $f(x)$ as x approaches a is L if $f(x)$ is arbitrarily close to the single, real number L whenever x is sufficiently close to, but less than, a. We write this as:

$$\lim_{x \to a^-} f(x) = L$$

or sometimes we write $f(x) \to L$ when $x \to a^-$. ◊

Definition 1.2.3 Right-Sided Limit. For the function $f(x)$ defined at all x-xalues around and greater than a, we say that the **right-sided limit of** $f(x)$ as x approaches a is L if $f(x)$ is arbitrarily close to the single, real number L whenever x is sufficiently close to, but greater than, a. We write this as:

$$\lim_{x \to a^+} f(x) = L$$

or sometimes we write $f(x) \to L$ when $x \to a^+$. ◊

Theorem 1.2.4 Mismatched Limits. *For a function $f(x)$, if both $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then we say that $\lim_{x \to a} f(x)$ does not exist. That is, there is no single real number L that $f(x)$ is arbitrarily close to for x-values that are sufficiently close to, but not equal to, $x = a$.*

1.2.2 Approximating Limits Using Our New Definition

Activity 1.2.2 Approximating Limits. For each of the following graphs of functions, approximate the limit in question. When you do so, approximate the values of the relevant one-sided limits as well.

(a) Approximate $\lim_{x \to 1} f(x)$ using the graph of the function $f(x)$ below.

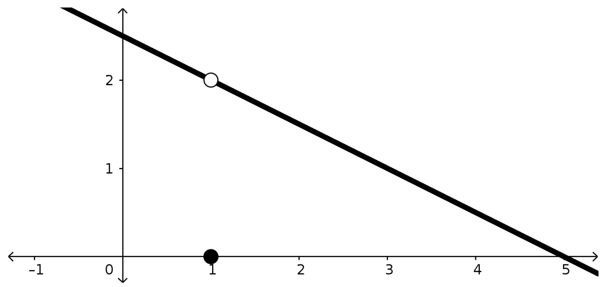


Figure 1.2.5

(b) Approximate $\lim_{x\to 2} g(x)$ using the graph of the function $g(x)$ below.

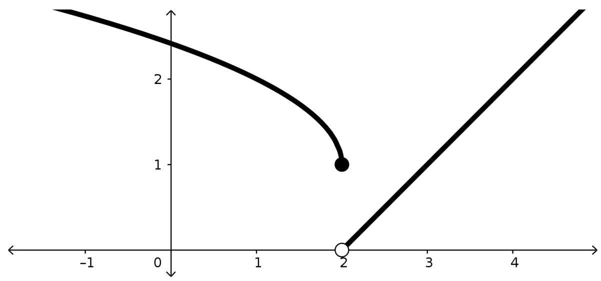


Figure 1.2.6

(c) Approximate the following three limits using the graph of the function $h(x)$ below.

- $\lim_{x\to -1} h(x)$
- $\lim_{x\to 0} h(x)$
- $\lim_{x\to 2} h(x)$

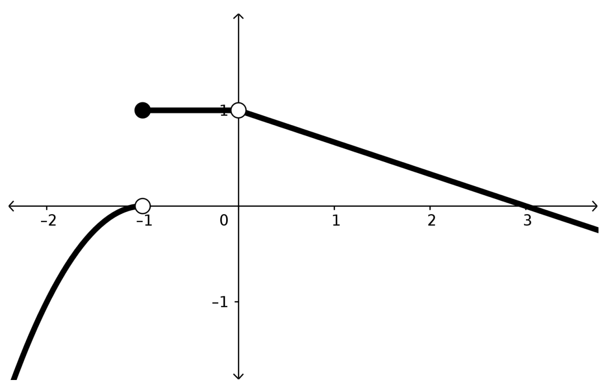


Figure 1.2.7

(d) Why do we say these are "approximations" or "estimations" of the limits we're interested in?

(e) Are there any limit statements that you made that you are 100% confident in? Which ones?

(f) Which limit statements are you least confident in? What about them makes them ones you aren't confident in?

(g) What extra details would you like to see to increase the confidence in your estimations? Are there changes we could make to the way these functions are represented that would make these approximations better or easier to make?

Activity 1.2.3 Approximating Limits Numerically. For each of the following tables of function values, approximate the limit in question. When you do so, approximate the values of the relevant one-sided limits as well.

(a) Approximate $\lim_{x \to 1} f(x)$ using the table of values of $f(x)$ below.

Table 1.2.8

x	0.5	0.9	0.99	1	1.01	1.1	1.5
$f(x)$	8.672	9.2	9.0001	-7	8.9998	9.5	7.59

(b) Approximate $\lim_{x \to -3} g(x)$ using the table of values of $g(x)$ below.

Table 1.2.9

x	-3.5	-3.1	-3.01	-3	-2.99	-2.9	-2.5
$g(x)$	-4.41	-3.89	-4.003	-4	7.035	2.06	-4.65

(c) Approximate $\lim_{x \to \pi} h(x)$ using the table of values of $h(x)$ below.

Table 1.2.10

x	3.1	3.14	3.141	π	3.142	3.15	3.2
$h(x)$	6	6	6	undefined	5.915	6.75	8.12

(d) Are you 100% confident about the existence (or lack of existence) of any of these limits?

(e) What extra details would you like to see to increase the confidence in your estimations? Are there changes we could make to the way these functions are represented that would make these approximations better or easier to make?

1.3 Evaluating Limits

1.3.1 Adding Precision to Our Estimations

Activity 1.3.1 From Estimating to Evaluating Limits (Part 1). Let's consider the following graphs of functions $f(x)$ and $g(x)$.

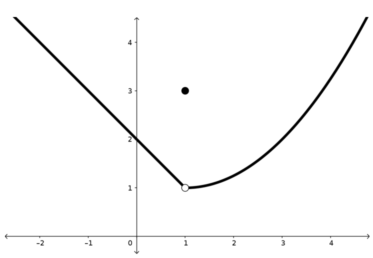


Figure 1.3.1 Graph of the function $f(x)$.

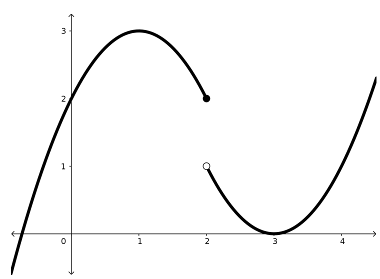


Figure 1.3.2 Graph of the function $g(x)$.

(a) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x \to 1^-} f(x)$

- $\lim_{x\to 1^+} f(x)$
- $\lim_{x\to 1} f(x)$

(b) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x\to 2^-} g(x)$
- $\lim_{x\to 2^+} g(x)$
- $\lim_{x\to 2} g(x)$

(c) Find the values of $f(1)$ and $g(2)$.

(d) For the limits and function values above, which of these are you most confident in? What about the limit, function value, or graph of the function makes you confident about your answer?

Similarly, which of these are you the least confident in? What about the limit, function value, or graph of the function makes you not have confidence in your answer?

Activity 1.3.2 From Estimating to Evaluating Limits (Part 2). Let's consider the following graphs of functions $f(x)$ and $g(x)$, now with the added labels of the equations defining each part of these functions.

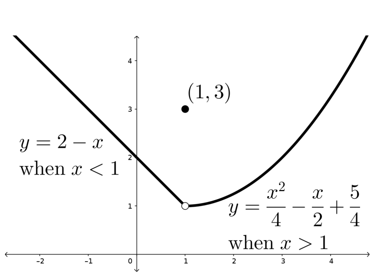


Figure 1.3.3 Graph of the function $f(x)$.

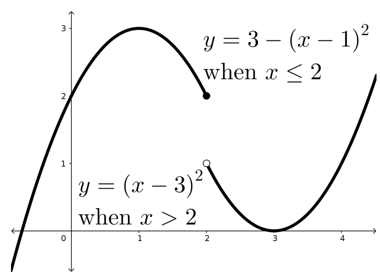


Figure 1.3.4 Graph of the function $g(x)$.

(a) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x\to 1^-} f(x)$
- $\lim_{x\to 1^+} f(x)$
- $\lim_{x\to 1} f(x)$

(b) Estimate the values of the following limits. If you believe that the limit does not exist, say so and explain why.

- $\lim_{x\to 2^-} g(x)$
- $\lim_{x\to 2^+} g(x)$
- $\lim_{x\to 2} g(x)$

(c) Does the addition of the function rules change the level of confidence you have in these answers? What limits are you more confident in with this added information?

(d) Consider these functions without their graphs:

$$f(x) = \begin{cases} 2 - x & \text{when } x < 1 \\ 3 & \text{when } x = 1 \\ \frac{x^2}{4} - \frac{x}{2} + \frac{5}{4} & \text{when } x > 1 \end{cases}$$

$$g(x) = \begin{cases} 3 - (x-1)^2 & \text{when } x \leq 2 \\ (x-3)^2 & \text{when } x > 2 \end{cases}$$

Find the limits $\lim_{x \to 1} f(x)$ and $\lim_{x \to 2} g(x)$. Compare these values of $f(1)$ and $g(2)$: are they related at all?

1.3.2 Limit Properties

Theorem 1.3.5 Combinations of Limits. *If $f(x)$ and $g(x)$ are two functions defined at x-values around, but maybe not at, $x = a$ and $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist, then we can evaluate limits of combinations of these functions.*

1. Sums: *The limit of the sum of $f(x)$ and $g(x)$ is the sum of the limits of $f(x)$ and $g(x)$:*

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

2. Differences: *The limit of a difference of $f(x)$ and $g(x)$ is the difference of the limits of $f(x)$ and $g(x)$:*

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

3. Coefficients: *If k is some real number coefficient, then:*

$$\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$$

4. Products: *The limit of a product of $f(x)$ and $g(x)$ is the product of the limits of $f(x)$ and $g(x)$:*

$$\lim_{x \to a} (f(x) \cdot g(x)) = \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} g(x)\right)$$

5. Quotients: *The limit of a quotient of $f(x)$ and $g(x)$ is the quotient of the limits of $f(x)$ and $g(x)$ (provided that you do not divide by 0):*

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\left(\lim_{x \to a} f(x)\right)}{\left(\lim_{x \to a} g(x)\right)} \qquad (\textit{if } \lim_{x \to a} g(x) \neq 0)$$

Theorem 1.3.6 Limits of Two Basic Functions. *Let a be some real number.*

1. Limit of a Constant Function: *If k is some real number constant, then:*

$$\lim_{x \to a} k = k$$

2. Limit of the Identity Function:

$$\lim_{x\to a} x = a$$

Activity 1.3.3 Limits of Polynomial Functions. We're going to use a combination of properties from Theorem 1.3.5 and Theorem 1.3.6 to think a bit more deeply about polynomial functions. Let's consider a polynomial function:

$$f(x) = 2x^4 - 4x^3 + \frac{x}{2} - 5$$

(a) We're going to evaluate the limit $\lim_{x\to 1} f(x)$. First, use the properties from Theorem 1.3.5 to re-write this limit as 4 different limits added or subtracted together.

Answer.

$$\lim_{x\to 1}(2x^4) - \lim_{x\to 1}(4x^3) + \lim_{x\to 1}\left(\frac{x}{2}\right) - \lim_{x\to 1} 5$$

(b) Now, for each of these limits, re-write them as products of things until you have only limits of constants and identity functions, as in Theorem 1.3.6. Evaluate your limits.

Hint.

$$2\left(\lim_{x\to 1} x\right)^4 - 4\left(\lim_{x\to 1} x\right)^3 + \frac{1}{2}\left(\lim_{x\to 1} x\right) - \lim_{x\to 1} 5$$

(c) Based on the definition of a limit (Definition 1.2.1), we normally say that $\lim_{x\to 1} f(x)$ is not dependent on the value of $f(1)$. Why do we say this?

(d) Compare the values of $\lim_{x\to 1} f(x)$ and $f(1)$. Why do these values feel connected?

(e) Come up with a new polynomial function: some combination of coefficients with x's raised to natural number exponents. Call your new polynomial function $g(x)$. Evaluate $\lim_{x\to -1} g(x)$ and compare the value to $g(-1)$. Explain why these values are the same.

(f) Explain why, for any polynomial function $p(x)$, the limit $\lim_{x\to a} p(x)$ is the same value as $p(a)$.

Theorem 1.3.7 Limits of Polynomials. *If $p(x)$ is a polynomial function and a is some real number, then:*

$$\lim_{x\to a} p(x) = p(a)$$

1.4 First Indeterminate Forms

Activity 1.4.1 Limits of (Slightly) Different Functions.

(a) Using the graph of $f(x)$ below, approximate $\lim_{x\to 1} f(x)$.

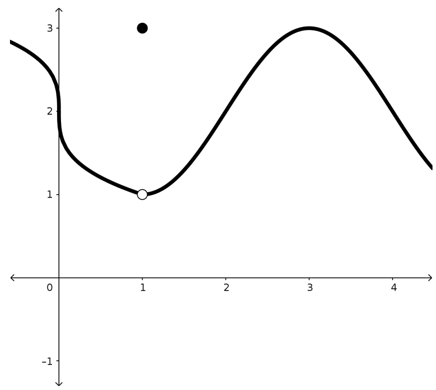


Figure 1.4.1

(b) Using the graph of the slightly different function $g(x)$ below, approximate $\lim_{x\to 1} g(x)$.

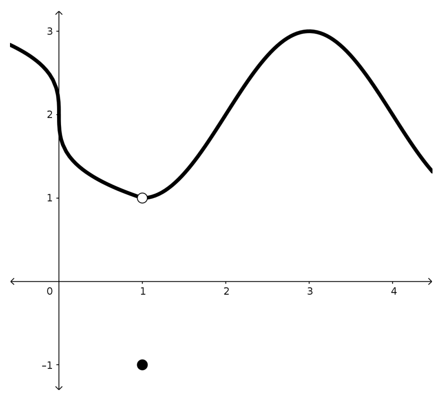


Figure 1.4.2

(c) Compare the values of $f(1)$ and $g(1)$ and discuss the impact that this difference had on the values of the limits.

(d) For the function $r(t)$ defined below, evaluate the limit $\lim_{x\to 4} r(t)$.

$$r(t) = \begin{cases} 2t - \frac{4}{t} & \text{when } t < 4 \\ 8 & \text{when } t = 4 \\ t^2 - t - 5 & \text{when } t > 4 \end{cases}$$

(e) For the slightly different function $s(t)$ defined below, evaluate the limit $\lim_{x\to 4} s(t)$.

$$s(t) = \begin{cases} 2t - \frac{4}{t} & \text{when } t \leq 4 \\ t^2 - t - 5 & \text{when } t > 4 \end{cases}$$

(f) Do the changes in the way that the function was defined impact the evaluation of the limit at all? Why not?

Theorem 1.4.3 Limits of (Slightly) Different Functions. *If $f(x)$ and $g(x)$ are two functions defined at x-values around a (but maybe not at $x = a$ itself) with $f(x) = g(x)$ for the x-values around a but with $f(a) \neq g(a)$ then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, if the limits exist.*

1.4.1 A First Introduction to Indeterminate Forms

Definition 1.4.4 Indeterminate Form. We say that a limit has an **indeterminate form** if the general structure of the limit could take on any different value, or not exist, depending on the specific circumstances.

For instance, if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then we say that the limit $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ has an indeterminate form. We typically denote this using the informal symbol $\frac{0}{0}$, as in:

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) \overset{?}{\to} \frac{0}{0}.$$

◊

Activity 1.4.2

(a) Were going to evaluate $\lim_{x\to 3}\left(\frac{x^2-7x+12}{x-3}\right)$.

- First, check that we get the indeterminate form $\frac{0}{0}$ when $x \to 3$.
- Now we want to find a new function that is equivalent to $f(x) = \frac{x^2-7x+12}{x-3}$ for all x-values other than $x = 3$. Try factoring the numerator, $x^2 - 7x + 12$. What do you notice?
- "Cancel" out any factors that show up in the numerator and denominator. Make a special note about what that factor is.
- This function is equivalent to $f(x) = \frac{x^2-7x+12}{x-3}$ except at $x = 3$. The difference is that this function has an actual function output at $x = 3$, while $f(x)$ doesn't. Evaluate the limit as $x \to 3$ for your new function.

(b) Now we'll evaluate a new limit: $\lim_{x\to 1}\left(\frac{\sqrt{x^2+3}-2}{x^2-5x+4}\right)$.

- First, check that we get the indeterminate form $\frac{0}{0}$ when $x \to 1$.
- Now we want a new function that is equivalent to $g(x) = \frac{\sqrt{x^2+3}-2}{x^2-5x+4}$ for all x-values other than $x = 1$. Try multiplying the numerator and the denominator by $(\sqrt{x^2+3}+2)$. We'll call this the "conjugate" of the numerator.
- In your multiplication, confirm that $(\sqrt{x^2+3}-2)(\sqrt{x^2+3}+2) = (x^2+3)-4$.
- Try to factor the new numerator and denominator. Do you notice anything? Can you "cancel" anything? Make another note of what factor(s) you cancel.
- This function is equivalent to $g(x) = \frac{\sqrt{x^2+3}-2}{x^2-5x+4}$ except at $x = 1$. The difference is that this function has an actual function output at $x = 1$, while $g(x)$ doesn't. Evaluate the limit as $x \to 1$ for your new function.

(c) Our last limit in this activity is going to be $\lim_{x\to -2}\left(\frac{3-\frac{3}{x+3}}{x^2+2x}\right)$.

- Again, check to see that we get the indeterminate form $\frac{0}{0}$ when $x \to -2$.
- Again, we want a new function that is equivalent to $h(x) = \dfrac{3 - \frac{3}{x+3}}{x^2 + 2x}$ for all x-values other than $x = -2$. Try completing the subtraction in the numerator, $3 - \frac{3}{x+3}$, using "common denominators."
- Try to factor the new numerator and denominator(s). Do you notice anything? Can you "cancel" anything? Make another note of what factor(s) you cancel.
- For the final time, we've found a function that is equivalent to $h(x) = \dfrac{3 - \frac{3}{x+3}}{x^2 + 2x}$ except at $x = -2$. The difference is that this function has an actual function output at $x = -2$, while $h(x)$ doesn't. Evaluate the limit as $x \to -2$ for your new function.

(d) In each of the previous limits, we ended up finding a factor that was shared in the numerator and denominator to cancel. Think back to each example and the factor you found. Why is it clear that these *must* have been the factors we found to cancel?

(e) Let's say we have some new function $f(x)$ where $\lim\limits_{x \to 5} f(x) \overset{?}{\to} \dfrac{0}{0}$. You know, based on these examples, that you're going to apply *some* algebra trick to re-write your function, factor, and cancel. Can you predict what you will end up looking for to cancel in the numerator and denominator? Why?

1.4.2 What if There Is No Algebra Trick?

We've seen some nice examples above where we were able to use some algebra to manipulate functions in such was as to force some shared factor in the numerator and denominator into revealing itself. From there, we were able to apply Theorem 1.4.3 and swap out our problematic function with a new one, knowing that the limit would be the same.

But what if we can't do that? What if the specific structure of the function seems *resistent* somehow to our attempts at wielding algebra?

This happens a lot, and we'll investigate some more of those types of limits in Section 4.9. For now, though, let's look at a very famous limit and reason our way through the indeterminate form.

Activity 1.4.3 Let's consider a new limit:

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}.$$

This one is strange!

(a) Notice that this function, $f(\theta) = \dfrac{\sin(\theta)}{\theta}$, is resistent to our algebra tricks:

- There's nothing to "factor" here, since our trigonometric function is not a polynomial.
- We can't use a trick like the "conjugate" to multiply and re-write, since there's no square roots and also only one term in the numerator.
- There aren't any fractions that we can combine by addition or subtraction.

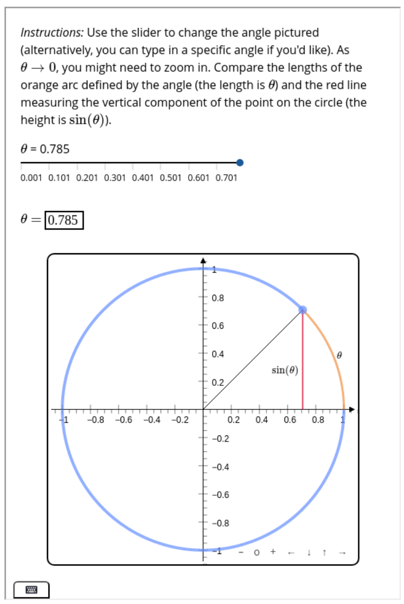
(b) Be frustrated at this new limit for resisting our algebra tricks.

(c) Now let's think about the meaning of $\sin(\theta)$ and even θ in general. In this text, we will often use Greek letters, like θ, to represent angles. In general, these angles will be measured in radians (unless otherwise specified). So what does the sine function *do* or *tell us*? What is a radian?

Hint 1. On the unit circle, if we plot some point at an angle of θ, then the coordinates of that point can be represented with trig functions! Which ones?

Hint 2. The length of the curve defining a unit circle is 2π. This also corresponds to the angle we would use to represent moving all the way around the circle. What must the length of the portion of the circle be up to some point at an angle θ?

(d) Let's visualize our limit, then, by comparing the length of the arc and the height of the point as $\theta \to 0$.



(e) Explain to yourself, until you are absolutely certain, why the two lengths *must* be the same in the limit as $\theta \to 0$. What does this mean about $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}$?

1.5 Limits Involving Infinity

Two types of limits involving infinity. In both cases, we'll mostly just consider what happens when we divide by small things and what happens when we divide by big things. We can summarize this here, though:

Fractions with small denominators are big, and fractions with big denominators are small.

1.5.1 Infinite Limits

Activity 1.5.1 What Happens When We Divide by 0? First, let's make sure we're clear on one thing: there is no real number than is represented as some other number divided by 0.

When we talk about "dividing by 0" here (and in Section 1.4), we're talking about the behavior of some function in a limit. We want to consider what it might look like to have a function that involves division where the denominator *gets arbitrarily close to 0* (or, the limit of the denominator is 0).

(a) Remember when, once upon a time, you learned that dividing one a number by a fraction is the same as multiplying the first number by the reciprocal of the fraction? Why is this true?

(b) What is the relationship between a number and its reciprocal? How does the size of a number impact the size of the reciprocal? Why?

(c) Consider $12 \div N$. What is the value of this division problem when:

- $N = 6$?
- $N = 4$?
- $N = 3$?
- $N = 2$?
- $N = 1$?

(d) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = \frac{1}{2}$?
- $N = \frac{1}{3}$?
- $N = \frac{1}{4}$?
- $N = \frac{1}{6}$?
- $N = \frac{1}{1000}$?

(e) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to 0^+$? Note that this means that the x-values we're considering most are very small and positive.

(f) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to 0^-$? Note that this means that the x-values we're considering most are very small and negative.

Definition 1.5.1 Infinite Limit. We say that a function $f(x)$ has an **infinite limit** at a if $f(x)$ is arbitrarily large (positive or negative) when x is sufficiently close to, but not equal to, $x = a$.

We would then say, depending on the sign of the values of $f(x)$, that:

$$\lim_{x \to a^-} f(x) = \pm\infty \qquad\qquad \lim_{x \to a^+} f(x) = \pm\infty.$$

If the sign of both one-sided limits are the same, we can say that $\lim_{x \to a} f(x) = \pm\infty$ (depending on the sign), but it is helpful to note that, by the definition of the Limit of a Function, this limit does not exist, since $f(x)$ is not arbitrarily close to a single real number. ◊

Theorem 1.5.2 Dividing by 0 in a Limit. *If* $f(x) = \dfrac{g(x)}{h(x)}$ *with* $\lim_{x \to a} g(x) \neq 0$ *and* $\lim_{x \to a} h(x) = 0$*, then* $f(x)$ *has an* *Infinite Limit* *at* a*. We will often denote this behavior as:*

$$\lim_{x \to a} f(x) \overset{?}{\to} \frac{\#}{0}$$

where # is meant to be some shorthand representation of a non-zero limit in the numerator (often, but not necessarily, some real number).

Evaluating Infinite Limits.

Once we know that $\lim_{x \to a} f(x) \overset{?}{\to} \dfrac{\#}{0}$, we know a bunch of information right away!

- This limit doesn't exist.
- The function $f(x)$ has a vertical asymptote at $x = a$, causing these unbounded y-values near $x = a$.
- The one sided limits *must* be either ∞ or $-\infty$.
- We only need to focus on the sign of the one sided limits! And signs of products and quotients are easy to follow.

So a pretty typical process is to factor as much as we can, and check the sign of each factor (in a numerator or denominator) as $x \to a^-$ and $x \to a^+$. From there, we can find the sign of $f(x)$ in both of those cases, which will tell us the one-sided limit.

Example 1.5.3 For each function, find the relevant one-sided limits at the input-value mentioned. If you can use a two-sided limit statement to discuss the behavior of the function around this input-value, then do so.

(a) $\left(\dfrac{2x^2 - 5x + 1}{x^2 + 8x + 16}\right)$ and $x = -4$

(b) $\left(\dfrac{4x^2 - x^5}{x^2 - 4x + 3}\right)$ and $x = 1$

(c) $\sec(\theta)$ and $\theta = \dfrac{\pi}{2}$

□

1.5.2 End Behavior Limits

Activity 1.5.2 What Happens When We Divide by Infinity? Again, we need to start by making something clear: if we were really going to try divide some real number by infinity, then we would need to re-build our definition of what it means to divide. In the context we're in right now, we only have division defined as an operation for real (and maybe complex) numbers. Since infinity is neither, then we will not literally divide by infinity.

When we talk about "dividing by infinity" here, we're again talking about the behavior of some function in a limit. We want to consider what it might look like to have a function that involves division where the denominator *gets arbitrarily large (positive or negative)* (or, the limit of the denominator is infinite).

(a) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = 1$?
- $N = 6$?
- $N = 12$?
- $N = 24$?
- $N = 1000$?

(b) Let's again consider $12 \div N$. What is the value of this division problem when:

- $N = -1$?
- $N = -6$?
- $N = -12$?
- $N = -24$?
- $N = -1000$?

(c) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to \infty$? Note that this means that the x-values we're considering most are very large and positive.

(d) Consider a function $f(x) = \frac{12}{x}$. What happens to the value of this function when $x \to -\infty$? Note that this means that the x-values we're considering most are very large and negative.

(e) Why is there no difference in the behavior of $f(x)$ as $x \to \infty$ compared to $x \to -\infty$ when the sign of the function outputs are opposite ($f(x) > 0$ when $x \to \infty$ and $f(x) < 0$ when $x \to -\infty$)?

Definition 1.5.4 Limit at Infinity. If $f(x)$ is defined for all large and positive x-values and $f(x)$ gets arbitrarily close to the single real number L when x gets sufficiently large, then we say:

$$\lim_{x \to \infty} f(x) = L.$$

Similarly, if $f(x)$ is defined for all large and negative x-values and $f(x)$ gets arbitrarily close to the single real number L when x gets sufficiently negative, then we say:

$$\lim_{x \to -\infty} f(x) = L.$$

In the case that $f(x)$ has a **limit at infinity** that exists, then we say $f(x)$ has a horizontal asymptote at $y = L$.

Lastly, if $f(x)$ is defined for all large and positive (or negative) x-values and $f(x)$ gets arbitrarily large and positive (or negative) when x gets sufficiently large (or negative), then we could say:

$$\lim_{x \to -\infty} f(x) = \pm\infty \text{ or } \lim_{x \to \infty} f(x) = \pm\infty.$$

◊

Because the primary focus for limits at infinity is the end behavior of a function, we will often refer to these limits as **end behavior limits**.

Theorem 1.5.5 End Behavior of Reciprocal Power Functions. *If p is a positive real number, then:*

$$\lim_{x\to\infty}\left(\frac{1}{x^p}\right) = 0 \text{ and } \lim_{x\to-\infty}\left(\frac{1}{x^p}\right) = 0.$$

Theorem 1.5.6 Polynomial End Behavior Limits. *For some polynomial function:*

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$$

with n a positive integer (the degree) and all of the coefficients $a_0, a_1, ..., a_n$ real numbers (with $a_n \neq 0$), then

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n$$

That is, the leading term (the term with the highest exponent) defines the end behavior for the whole polynomial function.

Proof. Consider the polynomial function:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$$

where n is some integer and a_k is a real number for $k = 0, 1, 2, ..., n$. For simplicity, we will consider only the limit as $x \to \infty$, but we could easily repeat this exact proof for the case where $x \to -\infty$.

Before we consider this limit, we can factor out x^n, the variable with the highest exponent:

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0 \\ &= x^n\left(\frac{a^n x^n}{x^n} + \frac{a_{n-1}x^{n-1}}{x^n} + ... + \frac{a_2x^2}{x^n} + \frac{a_1x}{x^n} + \frac{a_0}{x^n}\right) \\ &= x^n\left(a_n + \frac{a_{n-1}}{x} + ... + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \end{aligned}$$

Now consider the limit of this product:

$$\begin{aligned} \lim_{x\to\infty} p(x) &= \lim_{x\to\infty} x^n\left(a_n + \frac{a_{n-1}}{x} + ... + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \\ &= \left(\lim_{x\to\infty} x^n\right)\left(\lim_{x\to\infty} a_n + \frac{a_{n-1}}{x} + ... + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \end{aligned}$$

We can see that in the second limit, we have a single constant term, a_n, followed by reciprocal power functions. Then, due to Theorem 1.5.5, we know that the second limit will by a_n, since the reciprocal power functions will all approach 0.

$$\begin{aligned} \lim_{x\to\infty} p(x) &= \left(\lim_{x\to\infty} x^n\right)\left(\lim_{x\to\infty} a_n + \frac{a_{n-1}}{x} + ... + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \\ &= \left(\lim_{x\to\infty} x^n\right)(a_n + 0 + ... + 0 + 0 + 0) \\ &= \left(\lim_{x\to\infty} x^n\right)(a_n) \\ &= \lim_{x\to\infty} a_n x^n \end{aligned}$$

And so $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n$ as we claimed. ■

Example 1.5.7 For each function, find the limits as $x \to \infty$ and $x \to -\infty$.

(a) $\left(\dfrac{2x^2 - 5x + 1}{x^2 + 8x + 16}\right)$

(b) $\left(\dfrac{4x^2 - x^5}{x^2 - 4x + 3}\right)$

(c) $\dfrac{|x|}{3x}$

□

Activity 1.5.3 Matching the Limits.

(a) We're going to look at four graphs of functions, as well as a list of limit statements. Match the limit statements with the graphs that match that behavior. Note that is is possible for a limit to be relevant on more than one graph.

(b) Now consider these four function definitions. Using your knowledge of limits, as well as the matching you've already done, match the definitions of these four functions with the graphs that go with them, and then also the limits that are relevant. (These limits will already be matched with the graphs, so you don't need to do further work here).

1.6 The Squeeze Theorem

Activity 1.6.1 A Weird End Behavior Limit. In this activity, we're going to find the following limit:

$$\lim_{x\to\infty}\left(\frac{\sin^2(x)}{x^2+1}\right).$$

This limit is a bit weird, in that we really haven't looked at trigonometric functions that much. We're going to start by looking at a different limit in the hopes that we can eventually build towards this one.

(a) Consider, instead, the following limit:

$$\lim_{x\to\infty}\left(\frac{1}{x^2+1}\right).$$

Find the limit and connect the process or intuition behind it to at least one of the results from this text.

Hint 1. Start with Theorem 1.3.5 to think about the numerator and denominator separately.

Hint 2. Can you use Theorem 1.5.6 in the denominator?

Hint 3. Is Theorem 1.5.5 relevant?

(b) Let's put this limit aside and briefly talk about the sine function. What are some things to remember about this function? What should we know? How does it behave?

(c) What kinds of values doe we expect $\sin(x)$ to take on for different values of x?

$$\_\_\_\_\_\_ \leq \sin(x) \leq \_\_\_\_\_\_$$

(d) What happens when we square the sine function? What kinds of values can that take on?

$$\underline{\qquad\qquad} \leq \sin^2(x) \leq \underline{\qquad\qquad}$$

(e) Think back to our original goal: we wanted to know the end behavior of $\dfrac{\sin^2(x)}{x^2+1}$. Right now we have two bits of information:

- We know $\lim_{x\to\infty}\left(\dfrac{1}{x^2+1}\right)$.
- We know some information about the behavior of $\sin^2(x)$. Specifically, we have some bounds on its values.

Can we combine this information?

In your inequality above, multiply $\left(\frac{1}{x^2+1}\right)$ onto all three pieces of the inequality. Make sure you're convinced about the direction or order of the inequality and whether or not it changes with this multiplication.

$$\underbrace{\frac{\qquad\qquad}{x^2+1}}_{\text{call this } f(x)} \leq \frac{\sin^2(x)}{x^2+1} \leq \underbrace{\frac{\qquad\qquad}{x^2+1}}_{\text{call this } h(x)}$$

(f) For your functions $f(x)$ and $h(x)$, evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} h(x)$.

(g) What do you think this means about the limit we're interested in, $\lim_{x\to\infty}\left(\dfrac{\sin^2(x)}{x^2+1}\right)$?

Theorem 1.6.1 The Squeeze Theorem. *For some functions $f(x)$, $g(x)$, and $h(x)$ which are all defined and ordered $f(x) \leq g(x) \leq h(x)$ for x-values near $x = a$ (but not necessarily at $x = a$ itself), and for some real number L, if we know that*

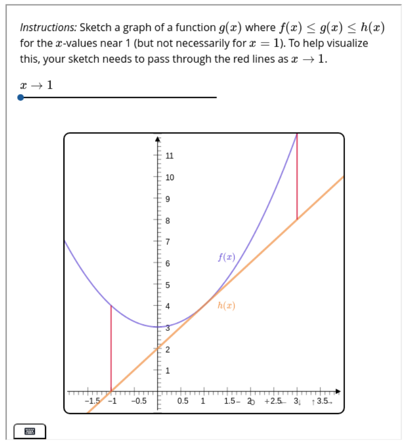
$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

then we also know that $\lim_{x\to a} g(x) = L$.

Activity 1.6.2 Sketch This Function Around This Point.

(a) Sketch or visualize the functions $f(x) = x^2 + 3$ and $h(x) = 2x + 2$, especially around $x = 1$.

(b) Now we want to add in a sketch of some function $g(x)$, all the while satisfying the requirements of the Squeeze Theorem.



(c) Use the Squeeze Theorem to evaluate and explain $\lim_{x\to 1} g(x)$ for your function $g(x)$.

(d) Is this limit dependent on the specific version of $g(x)$ that you sketched? Would this limit be different for someone else's choice of $g(x)$ given the same parameters?

(e) What information must be true (if anything) about $\lim_{x\to 3} g(x)$ and $\lim_{x\to 0} g(x)$?

Do we know that these limits exist? If they do, do we have information about their values?

1.7 Continuity and the Intermediate Value Theorem

1.7.1 Continuity as Connectedness

1.7.2 Continuity as Classification

Definition 1.7.1 Continuous at a Point. The function $f(x)$ is **continuous** at an x-value in the domain of $f(x)$ if $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

If $f(x)$ is not continuous at $x = a$, but one of the one-sided limits is equal to the function output, then we can define **directional continuity** at that point:

- We say $f(x)$ is **continuous on the left** at $x = a$ when $\lim_{x\to a^-} f(x) = f(a)$.
- We say $f(x)$ is **continuous on the right** at $x = a$ when $\lim_{x\to a^+} f(x) = f(a)$.

◊

Definition 1.7.2 Continuous on an Interval. We say that $f(x)$ is **continuous on the interval** (a, b) if $f(x)$ is continuous at every x-value with $a < x < b$.

If $f(x)$ is continuous on the right at $x = a$ and/or continuous on the left at $x = b$, then we will say that $f(x)$ is continuous on the interval $[a, b)$, $(a, b]$, or $[a, b]$, whichever is relevant. ◊

1.7.3 Discontinuities

Where is a Function not Continuous?

Most of the functions that we consider in this text will be continuous everywhere that it makes sense: on their domain. That is, if there is a point defined at some x-value, it is likely that the function's limit matches the y-value of the point. More specifically, though:

- A function is discontinuous at any location that results in an infinite limit. These are locations where $f(x)$ is undefined and the limit is infinite (and so doesn't exist).
- A function is, in general, discontinuous wherever it is undefined. This seems silly to say! We probably could have left this unsaid.
- A function that is defined as a piecewise function could be discontinuous at locations where the pieces meet: maybe the limit doesn't exist, or maybe the function value is not defined, or maybe the limit exists and the function value is defined but they do not match.

1.7.4 Intermediate Value Theorem

Theorem 1.7.3 Intermediate Value Theorem. *If $f(x)$ is a function that is continuous on $[a, b]$ with $f(a) \neq f(b)$ and L is any real number between $f(a)$ and $f(b)$ (either $f(a) < L < f(b)$ or $f(b) < L < f(a)$), then there exists some c between a and b ($a < c < b$) such that $f(c) = L$.*

This theorem was stated as early as the 5th century BCE by Bryson of Heraclea. Back then, a really interesting problem was related to "squaring the circle." That is, given a circle with some measurable radius, can we construct a square with equal area? This is obviously true, in that we can just use a square with the side length $r\sqrt{\pi}$. What we typically mean by "construct," though, is to create this square using only a compass and straightedge (a ruler without length markings) and only a finite number of steps. This was finally proven to be impossible in 1882, approximately 2300 years later.

Bryson of Heraclea knew that the square itself existed (even if he couldn't construct it) because he was able to find a circle with area less than the square (by inscribing a circle inside of the square) and a circle with area greater than the square (where the square is inscribed in the circle). Since he posited that he could increase the size of the circle in a continuous manner (without using those words), he claimed that a square with area equal to that of the circle must exist, since the area of the circle passes through all values from the smaller area to the larger area.

Chapter 2

Derivatives

2.1 Introduction to Derivatives

We'll start this off by thinking about slopes. Before we begin, you should be able to answer the following questions:

- What *is* a slope? How could you describe it?
- How do you calculate the slope of a line between two points?
- If we have a function $f(x)$ and we pick two points on the curve of the function, what does the slope of a straight line connecting the two points tell us? What kind of behavior about $f(x)$ does this slope describe?

2.1.1 Defining the Derivative

Activity 2.1.1 Thinking about Slopes. We're going to calculate and make some conjectures about slopes of lines between points, where the points are on the graph of a function. Let's define the following function:

$$f(x) = \frac{1}{x+2}.$$

(a) We're going to calculate a lot of slopes! Calculate the slope of the line connecting each pair of points on the curve of $f(x)$:

- $(-1, f(-1))$ and $(0, f(0))$
- $(-0.5, f(-0.5))$ and $(0, f(0))$
- $(-0.1, f(-0.1))$ and $(0, f(0))$
- $(-0.001, f(-.001))$ and $(0, f(0))$

(b) Let's calculate another group of slopes. Find the slope of the lines connecting these pairs of points:

- $(0, f(0))$ and $(1, f(1))$
- $(0, f(0))$ and $(0.5, f(0.5))$
- $(0, f(0))$ and $(0.1, f(0.1))$
- $(0, f(0))$ and $(0.001, f(0.001))$

(c) Just to make it clear what we've done, lay out your slopes in this table:

Between $(0, f(0))$ **and...**	**Slope**
$(1, f(1))$	
$(0.5, f(0.5))$	
$(0.1, f(0.1))$	
$(0.01, f(0.01))$	
$(-0.01, f(-0.01))$	
$(-0.1, f(-0.1))$	
$(-0.5, f(-0.5))$	
$(-1, f(-1))$	

(d) Now imagine a line that is tangent to the graph of $f(x)$ at $x = 0$. We are thinking of a line that touches the graph at $x = 0$, but runs along side of the curve there instead of through it.

Make a conjecture about the slope of this line, using what we've seen above.

(e) Can you represent the slope you're thinking of above with a limit? What limit are we approximating in the slope calculations above? Set up the limit and evaluate it, confirming your conjecture.

Activity 2.1.2 Finding a Tangent Line. Let's think about a new function, $g(x) = \sqrt{2 - x}$. We're going to think about this function around the point at $x = 1$.

(a) Ok, we are going to think about this function at this point, so let's find the coordinates of the point first. What's the y-value on our curve at $x = 1$?

(b) Use a limit similar to the one you constructed in Activity 2.1.1 to find the slope of the line tangent to the graph of $g(x)$ at $x = 1$.

(c) Now that you have a slope of this line, and the coordinates of a point that the line passes through, can you find the equation of the line?

Definition 2.1.1 Derivative at a Point. For a function $f(x)$, we say that the **derivative** of $f(x)$ at $x = a$ is:

$$f'(a) = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right)$$

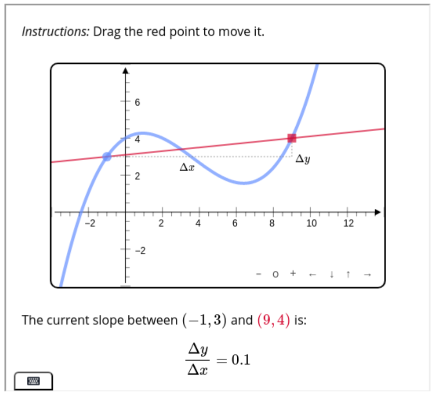
provided that the limit exists.

If $f'(a)$ exists, then we say $f(x)$ is **differentiable** at a. ◊

We can investigate this definition visually. Consider the function $f(x)$ plotted below, where we will look at the point $(-1, f(-1))$. In the definition of the limit, we'll let $a = -1$, and so consider:

$$\lim_{x \to -1} \left(\frac{f(x) - f(-1)}{x - (-1)} \right).$$

Can you estimate the limit of the slope of the tangent line as $x \to -1$?



Does it look like the limit of the slope between $(-1, f(-1))$ and $(x, f(x))$ exists as $x \to -1$? What do you think it is?

2.1.2 Calculating a Bunch of Slopes at Once

Activity 2.1.3 Calculating a Bunch of Slopes. Let's do this all again, but this time we'll calculate the slope at a bunch of different points on the same function.

Let's use $j(x) = x^2 - 4$.

(a) Start calculating the following derivatives, using the definition of the Derivative at a Point:

- $j'(-2)$
- $j'(0)$
- $j'(1/3)$
- $j'(-1)$

(b) Stop calculating the above derivatives when you get tired/bored of it. How many did you get through?

(c) Notice how repetitive this is: on one hand, we have to set up a completely different limit each time (since we're looking at a different point on the function each time). On the other hand, you might have noticed that the work is all the same: you factor and cancel over and over. These limits are all ones that we covered in Section 1.4 First Indeterminate Forms, and so it's no surprise that we keep using the same algebra manipulations over and over again to evaluate these limits.

Do you notice any patterns, any connections between the x-value you used for each point and the slope you calculated at that point? You might need to go back and do some more.

(d) Try to evaluate this limit in general:

$$\begin{aligned} j'(a) &= \lim_{x \to a} \left(\frac{j(x) - j(a)}{x - a} \right) \\ &= \lim_{x \to a} \left(\frac{(x^2 - 4) - (a^2 - 4)}{x - a} \right). \end{aligned}$$

Remember, you know how this goes! You're going to do the same sorts of algebra that you did earlier!

What is the formula, the pattern, the way of finding the slope on the $j(x)$ function at any x-value, $x = a$?

(e) Confirm this by using your new formula to re-calculate the following derivatives:

- $j'(-2)$
- $j'(0)$
- $j'(1/3)$
- $j'(-1)$

We're going to try to think about the derivative as something that can be calculated in general, as well as something that can be calculated at a point. We'll define a new way of calculating it, still a limit of slopes, that will be a bit more general.

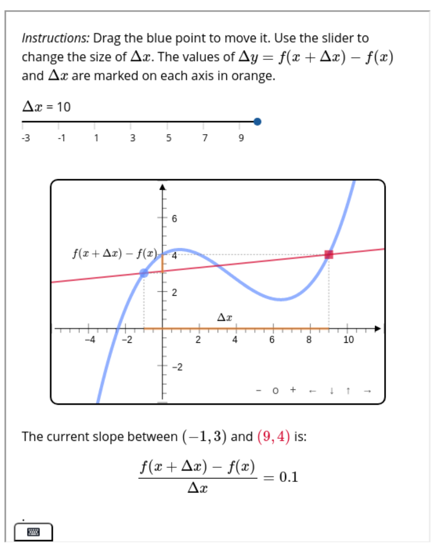
Definition 2.1.2 The Derivative Function. For a function $f(x)$, the derivative of $f(x)$, denoted $f'(x)$, is:

$$f'(x) = \lim_{\Delta x \to 0} \left(\frac{f(x + \Delta x) - f(x)}{\Delta x} \right)$$

for x-values in the domain of $f(x)$ where this limit exists. ◊

This definition feels pretty different, but we can hopefully notice that this is really just calculating a slope. Notice, in the following plot, that there is a significant difference. In the visualization of the Derivative at a Point, the first point was fixed into place and the second point was the one that we moved and changed. It was the one with the variable x-value.

Notice in the following visualization that the *first* point is the one that is moveable while the *second* point is defined based on the first one (and the horizontal difference between the points, Δx). This means that we don't need to define one specific point, and can find the slope of the line tangent to $f(x)$ at some changing x-value.



2.2 Interpreting Derivatives

What is a derivative?

This can feel like a silly question, since we're calculating it and getting used to finding them. But what is it?

In this section, we just want to remind ourselves of what this object is, how we should hold it in our minds as we move through the course, and then practice being flexible with this interpretation.

2.2.1 The Derivative is a Slope

Activity 2.2.1 Interpreting the Derivative as a Slope. In Activity 2.1.1 Thinking about Slopes and Activity 2.1.2 Finding a Tangent Line, we built the idea of a derivative by calculating slopes and using them. Let's continue this by considering the function $f(x) = \frac{1}{x^2}$.

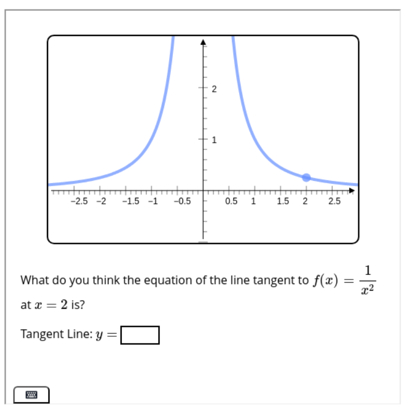
(a) Use Definition 2.1.1 Derivative at a Point to find $f'(2)$. What does this value represent?

(b) We want to plot the line that would be tangent to the graph of $f(x)$ at $x = 2$.

Remember that we can write the equation of a line in two ways:

- The equation of a line with slope m that passes through the point $(a, f(a))$ is:
$$y = m(x - a) + f(a).$$
- The equation of a line with slope m that passes the point $(0, b)$ (this is another way of saying that the y-intercept of the line is b) is:
$$y = mx + b.$$

Find the equation of the line tangent to $f(x)$ at $x = 2$. Add it to the graph of $f(x) = \frac{1}{x^2}$ below to check your equation.



(c) This tangent line is very similar to the actual curve of the function $f(x)$ near $x = 2$. Another way of saying this is that while the slope of $f(x)$ is not always the value you found for $f'(2)$, it is close to that for x-values near 2.

Use this idea of slope to predict where the y-value of our function will be at 2.01.

Hint. We know that slope is $\frac{\Delta y}{\Delta x}$ and we're using the fact that $\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$ for small values of Δx.

Here, we have $\Delta x = 0.01$, so can you use the slope to approximate the corresponding Δy and figure out the new y-value?

(d) Compare this value with $f(2.01) = \frac{1}{2.01^2}$. How close was it?

2.2.2 The Derivative is a Rate of Change

Activity 2.2.2 Interpreting the Derivative as a Rate of Change. This is going to somewhat feel redundant, since maybe we know that a slope is really just a rate of change. But hopefully we'll be able to explore this a bit more and see how we can use a derivative to tell us information about some specific context.

Let's say that we want to model the speed of a car as it races along a strip of the road. By the time we start measuring it (we'll call this time 0), the position the car (along the straight strip of road) is:

$$s(t) = 73t + t^2,$$

where t is time measured in seconds and $s(t)$ is the position measured in feet. Let's say that this function is only relevant on the domain $0 \leq t \leq 15$. That is, we only model the position of the car for a 15-second window as it speeds past us.

(a) How far does the car travel in the 15 seconds that we model it? What was the car's average velocity on those 15 seconds?

(b) Calculate $s'(t)$, the derivative of $s(t)$, using Definition 2.1.2 The Derivative Function. What information does this tell us about our vehicle?

Hint. What is the rate at which the position (in feet) of the vehicle changes per unit time (in seconds)? What do we call this, and what are the units?

(c) Calculate $s'(0)$. Why is this smaller than the average velocity you found? What does that mean about the velocity of the car?

(d) If we call $v(t) = s'(t)$, then calculate $v'(t)$. Note that this is a derivative of a derivative.

(e) Find $v'(0)$. Why does this make sense when we think about the difference between the average velocity on the time interval and the value of $v(0)$ that we calculated?

(f) What does it mean when we notice that $v'(t)$ is constant? Explain this by interpreting it in terms of both the velocity of the vehicle as well as the position.

2.2.3 The Derivative is a Limit

Look back at the definition of Derivative at a Point. The end of it is interesting: "provided that the limit exists." We need to keep in mind that this is a limit, and so a derivative exists or fails to exist whenever that limit exists or fails to exist.

What are some ways that a limit fails to exist?

- A limit doesn't exist if the left-side limit and the right-side limit do not match: Theorem 1.2.4 Mismatched Limits.
- A limit doesn't exist if it is an Infinite Limit.

What do each of these situations look like when we're considering the limit of slopes?

When Does a Derivative Not Exist?

1. A derivative doesn't exist at points where the slopes on either side of the point don't match.
2. A derivative doesn't exist at points with vertical tangent lines.
3. A derivative doesn't exist at points where the function is not continuous.

2.2.4 The Derivative is a Function

Activity 2.2.3 Interpreting the Derivative as a Function. In Activity 2.1.3 Calculating a Bunch of Slopes, we calculated the derivative function for $j(x) = x^2 - 4$. Using the definition of The Derivative Function, we can see that $j'(x) = 2x$. Let's explore that a bit more.

(a) Sketch the graphs of $j(x) = x^2 - 4$ and $j'(x) = 2x$. Describe the shapes of these graphs.

(b) Find the coordinates of the point at $x = \frac{1}{2}$ on both the graph of $j(x)$ and $j'(x)$. Plot the point on each graph.

(c) Think back to our previous interpretations of the derivative: how do we interpret the y-value output you found for the j' function?

(d) Find the coordinates of another point at some other x-value on both the graph of $j(x)$ and $j'(x)$. Plot the point on each graph, and explain what the output of j' tells us at this point.

(e) Use your graph of $j'(x)$ to find the x-intercept of $j'(x)$. Locate the point on $j(x)$ with this same x-value. How do we know, visually, that this point is the x-intercept of $j'(x)$?

(f) Use your graph of $j'(x)$ to find where $j'(x)$ is positive. Pick two x-values where $j'(x) > 0$. What do you expect this to look like on the graph of $j(x)$? Find the matching points (at the same x-values) on the graph of $j(x)$ to confirm.

(g) Use your graph of $j'(x)$ to find where $j'(x)$ is negative. Pick two x-values where $j'(x) < 0$. What do you expect this to look like on the graph of $j(x)$? Find the matching points (at the same x-values) on the graph of $j(x)$ to confirm.

(h) Let's wrap this up with one final pair of points. Let's think about the point $(-3, 5)$ on the graph of $j(x)$ and the point $(-3, -6)$ on the graph of $j'(x)$. First, explain what the value of -6 (the output of j' at $x = -3$) means about the point $(-3, 5)$ on $j(x)$. Finally, why can we not use the value 5 (the output of j at $x = -3$) means about the point $(-3, -6)$ on $j'(x)$?

2.2.5 Notation for Derivatives

So far we've been using the "prime" notation to represent derivatives: the derivative of $f(x)$ is $f'(x)$. We will continue to use this notation, but we'll introduce a bunch of other ways of writing notation to represent the derivative.

Each new notation will emphasize some aspect of the derivative that will serve to be useful, even though they all represent essentially the same thing.

Function	**Derivative**	**Derivative at $x = a$**	**Emphasis**
$f(x)$	$f'(x)$	$f'(a)$	The derivative is a function. The function takes in x-value inputs and returns the slope of f at that x-value.
y	y'	$y'\Big\vert_{x=a}$	We can find slopes on any curve, not just functions. This is sometimes also used as a way to simplify the notation, especially when we want to manipulate equations involving y'.
y	$\frac{dy}{dx}$	$\frac{dy}{dx}\Big\vert_{x=a}$	The derivative is a slope. It is $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$, and we use dx and dy (called **differentials**) to represent Δx and Δy as the limits as $\Delta x \to 0$. This notation is also useful to tell us what the rate of change is: what is changing (in this case y) and what is it changing based on (in this case x).
$f(x)$	$\frac{d}{dx}(f(x))$	$\frac{d}{dx}(f(x))\Big\vert_{x=a}$	The derivative is an action that we do to some function. We can call it an **operator**, although we won't formally define that term in this text. We'll look at this idea more in Section 3.1. We can specify what we are expecting the input variable to be, based on the differential dx in the denominator.

2.3 Some Early Derivative Rules

We are going to break this topic into two parts:

1. We will try to find some common patterns or connections between derivatives and specific functions. For instance, when we use Definition 2.1.2 The Derivative Function to build a derivative, are there patterns in the work of evaluating that limit that will allow us to get through the limit work quickly? Can we group some functions together based on how we might deal with the limit?

2. We will try to think about derivatives a bit more generally and show how we can build some basic properties to help us think about differentiating variations of the functions that we recognize.

2.3.1 Derivatives of Common Functions

Activity 2.3.1 Derivatives of Power Functions. We're going to do a bit of pattern recognition here, which means that we will need to differentiate several different power functions. For our reference, a power function (in general) is a function in the form $f(x) = a(x^n)$ where n and a are real numbers, and $a \neq 0$.

Let's begin our focus on the power functions x^2, x^3, and x^4. We're going to use Definition 2.1.2 The Derivative Function a lot, so feel free to review it before we begin.

(a) Find $\frac{d}{dx}\left(x^2\right)$. As a brief follow up, compare this to the derivative $j'(x)$ that you found in Activity 2.1.3 Calculating a Bunch of Slopes. Why are they the same? What does the difference, the -4, in the $j(x)$ function do to the graph of it (compared to the graph of x^2) and why does this not impact the derivative?

Hint. Remember that the graph of $x^2 - 4$ has the same shape as the graph of x^2, but has been shifted down by 4 units. Why does this vertical shift not change the value of the derivative at any x-value?

(b) Find $\frac{d}{dx}\left(x^3\right)$.

Hint. Remember that $(x + \Delta x)^3 = (x + \Delta x)(x + \Delta x)(x + \Delta x)$

(c) Find $\frac{d}{dx}\left(x^4\right)$.

Hint. Remember that $(x+\Delta x)^4 = (x+\Delta x)(x+\Delta x)(x+\Delta x)(x+\Delta x)$

(d) Notice that in these derivative calculations, the main work is in multiplying $(x+\Delta x)^n$. Look back at the work done in all three of these derivative calculations and find some unifying steps to describe how you evaluate the limit/calculate the derivative *after* this tedious multiplication was finished. What steps did you do? Is it always the same thing?

Another way of stating this is: if I told you that I knew what $(x + \Delta x)^5$ was, could you give me some details on how the derivative limit would be finished?

(e) Finish the following derivative calculation:

$$\begin{aligned}\frac{d}{dx}\left(x^5\right) &= \lim_{\Delta x \to 0}\left(\frac{(x + \Delta x)^5 - x^5}{\Delta x}\right) \\ &= \lim_{\Delta x \to 0}\left(\frac{(x^5 + 5x^4\Delta x + 10x^3\Delta x^2 + 10x^2\Delta x^3 + 5x\Delta x^4 + \Delta x^5) - x^5}{\Delta x}\right) \\ &= \rightsquigarrow \ldots\end{aligned}$$

(f) Make a conjecture about the derivative of a power function in general, $\frac{d}{dx}(x^n)$.

Something to notice here is that the calculation in this limit is really dependent on knowing what $(x+\Delta x)^n$ is. When n is an integer with $n \geq 2$, this really just translates to multiplication. If we can figure out how to multiply $(x + \Delta x)^n$ in general, then this limit calculation will be pretty easy to do. We noticed that:

1. The first term of that multiplication will combine with the subtraction of x^n in the numerator and subtract to 0.

2. The rest of the terms in the multiplication have at least one copy of Δx, and so we can factor out Δx and "cancel" it with the Δx in the denominator.

3. Once this has done, we've escaped the portion of the limit that was giving us the $\frac{0}{0}$ indeterminate form, and so we can evaluate the limit as $\Delta x \to 0$. The result is just that whatever terms still have at least one remaining copy of Δx in it "go to" 0, and we're left with just the terms that do not have any copies of Δx in them.

Triangle binomial theorem for coefficients.

Theorem 2.3.1 Power Rule for Derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

We have shown that this is true for $n = 2, 3, 4, ...$, but this is also true for *any* value of n (including $n = 1$, non-integers, and non-positives). We will prove this more formally later (in Section 3.3), and until then we will be free to use this result.

Example 2.3.2 Let's confirm this Power Rule for two examples that we are familiar with.

(a) Find the derivative $\frac{d}{dx}(\sqrt{x})$ using the limit definition of the derivative function. Note that $\sqrt{x} = x^{1/2}$ and $\frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$.

(b) Find the derivative $\frac{d}{dx}\left(\frac{1}{x}\right)$ using the limit definition of the derivative function. Note that $\frac{1}{x} = x^{-1}$ and $-\frac{1}{x^2} = -x^{-2}$.

□

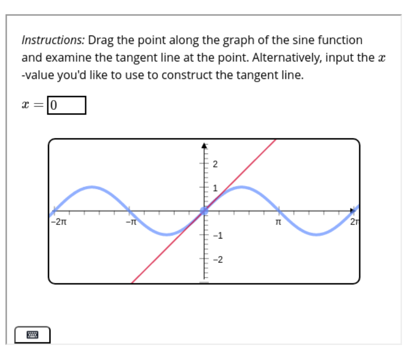
In this activity, we also found one other result.

Theorem 2.3.3 Derivative of a Constant Function. *If $y = k$ where k is some real number constant, then $y' = 0$. Another way of saying this is:*

$$\frac{d}{dx}(k) = 0.$$

Activity 2.3.2 Derivatives of Trigonometric Functions. Let's try to think through the derivatives of $y = \sin(\theta)$ and $y = \cos(\theta)$. In this activity, we'll look at graphs and try to collect some information about the derivative functions. We'll be practicing out interpretations, so if you need to brush up on Section 2.2 before we start, that's fine!

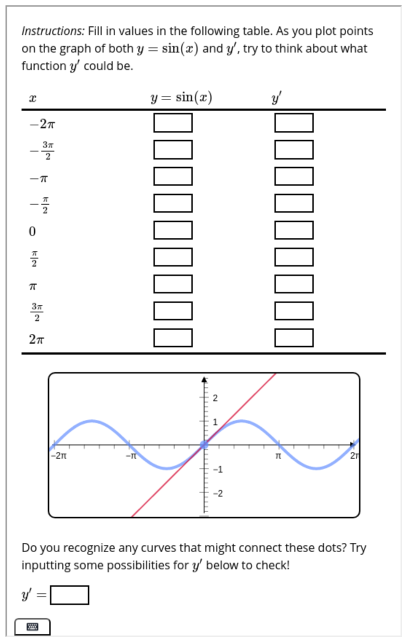
(a) The following plot includes both the graph of $y = \sin(x)$, and the line tangent to $y = \sin(x)$. Watch as the point where we build the tangent line moves along the graph, between $x = -2\pi$ and $x = 2\pi$.



Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

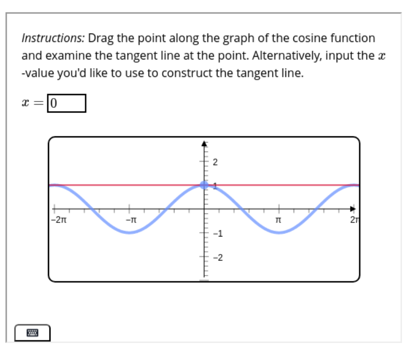
Hint. What kinds of values do the slopes take? Are there some values that these slopes will never be? Can you find any special points on this graph where you can actually tell what the slope is?

(b) We're going to get more specific here: let's find the coordinates of points that are on both the graph of $y = \sin(x)$ and it's derivative y'. Remember, to get the values for y', we're really looking at the slope of the tangent line at that point.



(c) Let's repeat this process using the $y = \cos(x)$ function instead.

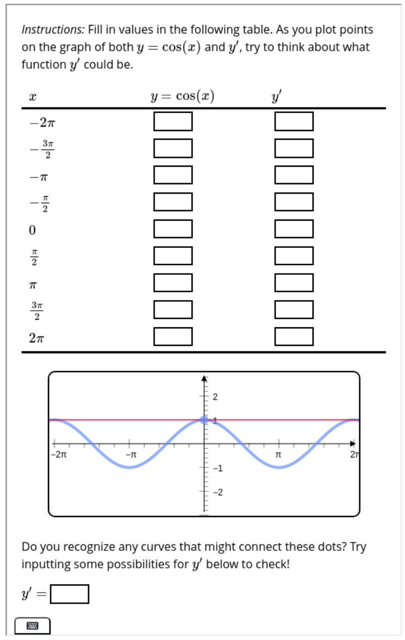
The following plot includes both the graph of $y = \cos(x)$, and the line tangent to $y = \cos(x)$. Watch as the point where we build the tangent line moves along the graph, between $x = -2\pi$ and $x = 2\pi$.



Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

Hint. What kinds of values do the slopes take? Are there some values that these slopes will never be? Can you find any special points on this graph where you can actually tell what the slope is?

(d) We're going to get more specific here: let's find the coordinates of points that are on both the graph of $y = \cos(x)$ and it's derivative y'. Remember, to get the values for y', we're really looking at the slope of the tangent line at that point.



Theorem 2.3.4 Derivatives of the Sine and Cosine Functions.

$$\frac{d}{d\theta}(\sin(\theta)) = \cos(\theta)$$

$$\frac{d}{d\theta}(\cos(\theta)) = -\sin(\theta)$$

Proof. In order to show why $\frac{d}{d\theta}(\sin(\theta)) = \cos(\theta)$ and $\frac{d}{dx}(\cos(\theta)) = -\sin(\theta)$, we will work with the limit definitions of both. Consider both:

$$\frac{d}{d\theta}(\sin(\theta)) = \lim_{\Delta\theta \to 0}\left(\frac{\sin(\theta + \Delta\theta) - \sin(\theta)}{\Delta\theta}\right)$$
$$\frac{d}{d\theta}(\cos(\theta)) = \lim_{\Delta\theta \to 0}\left(\frac{\cos(\theta + \Delta\theta) - \cos(\theta)}{\Delta\theta}\right)$$

Our goal is to re-write the numerators in both of these limits as something more usable. So far, we have been evaluating these types of limits (First Indeterminate Forms) using some algebraic manipulations. Instead of using algebra, we will use geometry.

Consider the unit circle below. We have plotted the angle θ and are reminded that the point on the circle that corresponds with the terminal side of the angle θ has coordinates $(\cos(\theta), \sin(\theta))$. We can label the sides of the triangle pictured below.

Now we consider a second point on the circle, this one formed by the terminal side of the angle $(\theta + \Delta\theta)$. This point has coordinates $(\cos(\theta + \Delta\theta), \sin(\theta + \Delta\theta))$. Note, below, that we want to find expressions for $\sin(\theta + \Delta\theta) - \sin(\theta)$ and $\cos(\theta + \Delta\theta) - \cos(\theta)$. We can find these geometrically.

Note, then, that the two triangles look to be similar triangles. In fact, we will find that in the limit as $\Delta\theta \to 0$, the length h matches the arc length $\Delta\theta$ perfectly, and thus lays at a right angle to the terminal side of the angle $\theta + \Delta\theta$.

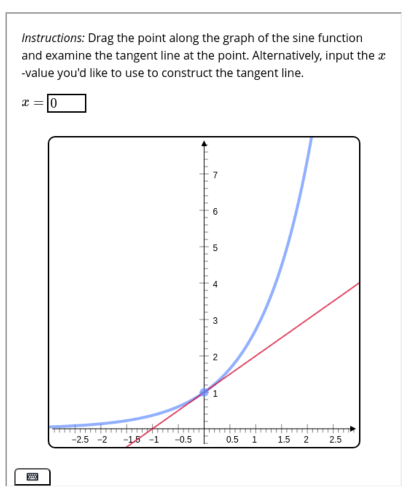
Since as $\Delta\theta \to 0$ we have $h \to \Delta\theta$, we can find the other side lengths as well: $(\sin(\theta + \Delta\theta) - \sin(\theta)) \to \Delta\theta \cos\theta$ and $(\cos(\theta + \Delta\theta) - \cos(\theta)) \to \Delta\theta \sin\theta$. So then $(\cos(\theta + \Delta\theta) - \cos(\theta)) \to -\Delta\theta \sin\theta$.
Consider, then, the limits involved in the derivative calculations that we built earlier.

$$\begin{aligned}
\frac{d}{d\theta}(\sin(\theta)) &= \lim_{\Delta\theta \to 0}\left(\frac{\sin(\theta+\Delta\theta)-\sin(\theta)}{\Delta\theta}\right)\\
&= \lim_{\Delta\theta \to 0}\left(\frac{\Delta\theta\cos(\theta)}{\Delta\theta}\right)\\
&= \lim_{\Delta\theta \to 0}(\cos(\theta))\\
&= \cos(\theta)\\
\frac{d}{d\theta}(\cos(\theta)) &= \lim_{\Delta\theta \to 0}\left(\frac{\cos(\theta+\Delta\theta)-\cos(\theta)}{\Delta\theta}\right)\\
&= \lim_{\Delta\theta \to 0}\left(\frac{-(\cos(\theta)-\cos(\theta+\Delta\theta))}{\Delta\theta}\right)\\
&= \lim_{\Delta\theta \to 0}\left(\frac{-\Delta\sin(\theta)}{\Delta\theta}\right)\\
&= \lim_{\Delta\theta \to 0}(-\sin(\theta))\\
&= -\sin(\theta)
\end{aligned}$$

So we have shown that $\frac{d}{d\theta}(\sin(\theta)) = \cos(\theta)$ and $\frac{d}{d\theta}(\cos(\theta)) = -\sin(\theta)$ as we claimed. ■

Activity 2.3.3 Derivative of the Exponential Function. We're going to consider a maybe-unfamiliar function, $f(x) = e^x$. We'll explore this function in a similar way to use thinking about the derivatives of sine and cosine in Activity 2.3.2: we'll look at a tangent line at different points, and think about the slope.

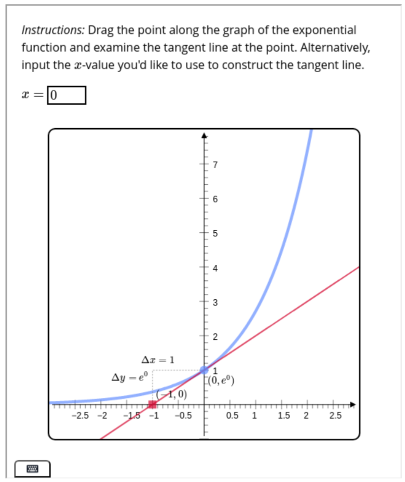
(a) The plot below includes both the graph of $y = e^x$ and the line tangent to $y = e^x$. Watch as the point moves along the curve.



Collect as much information about the derivative, y', as you can. What do you know about it? What are some facts about the slopes of the tangent lines in this animation?

Hint. Are there any x-values where the slope is negative? Are there any where the slope is equal to 0? What happens to the slopes as x increases?

(b) There is a slightly hidden fact about slopes and tangent lines in this animation. In the following animation, we'll add (and label) one more point. Let's look at this again, this time noting the point at which this tangent line crosses the x-axis. This will make it easier to think about slopes!



What information does this reveal about the slopes?

Hint. Especially it might be helpful to think about the slopes and their relationship to the y-value of the point we are building the tangent line at.

(c) Make a conjecture about the slope of the line tangent to the exponential function $y = e^x$ at any x-value. What do you believe the formula/equation for y' is then?

Theorem 2.3.5 Derivative of the Exponential Function.

$$\frac{d}{dx}\left(e^x\right) = e^x$$

2.3.2 Some Properties of Derivatives in General

Theorem 2.3.6 Combinations of Derivatives. *If $f(x)$ and $g(x)$ are differentiable functions, then the following statements about their derivatives are true.*

1. Sums: *The derivative of the sum of $f(x)$ and $g(x)$ is the sum of the derivatives of $f(x)$ and $g(x)$:*

$$\begin{aligned}\frac{d}{dx}\left(f(x) + g(x)\right) &= \left(\frac{d}{dx}f(x)\right) + \left(\frac{d}{dx}g(x)\right)\\ &= f'(x) + g'(x)\end{aligned}$$

2. Differences: *The derivative of the difference of $f(x)$ and $g(x)$ is the difference of the derivatives of $f(x)$ and $g(x)$:*

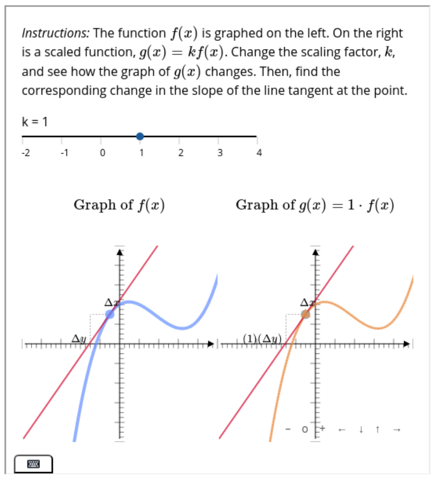
$$\frac{d}{dx}\left(f(x) - g(x)\right) = \left(\frac{d}{dx}f(x)\right) - \left(\frac{d}{dx}g(x)\right)$$

$$= f'(x) - g'(x)$$

3. Coefficients: *If k is some real number coefficient, then:*

$$\frac{d}{dx}\left(kf(x)\right) = k\left(\frac{d}{dx}f(x)\right)$$
$$= kf'(x)$$

We can think about each of these properties through the lense of how combining these functions impacts the slopes. For instance, if we wanted to visualize the property about coefficients (that $\frac{d}{dx}\left(kf(x)\right) = k\frac{d}{dx}\left(f(x)\right)$), we can visualize this coefficient as a scaling factor.



Example 2.3.7 Putting These Together. Find the following derivatives:

(a) $\frac{d}{dx}\left(4x^5 - \frac{5x}{2} + 6\cos(x) - 1\right)$

Solution.

$$\frac{d}{dx}\left(4x^5 - \frac{5x}{2} + 6\cos(x) - 1\right) = \frac{d}{dx}\left(4x^5\right) - \frac{d}{dx}\left(\frac{5x}{2}\right) + \frac{d}{dx}\left(6\cos(x)\right) - \frac{d}{dx}(1)$$
$$= 4\frac{d}{dx}\left(x^5\right) - \frac{5}{2}\frac{d}{dx}(x) + 6\frac{d}{dx}\left(\cos(x)\right) - \frac{d}{dx}(1)$$
$$= 4(5x^4) - \frac{5}{2}(1) + 6(-\sin(x)) - 0$$
$$= 20x^4 - \frac{5}{2} - 6\sin(x)$$

□

2.4 The Product and Quotient Rules

We saw in Theorem 2.3.6 Combinations of Derivatives that when we want to find the derivative of a sum or difference of functions, we can just find the derivatives of each function separately, and then combine the derivatives back together (by adding or subtracting). This, hopefully, is pretty intuitive: of course a slope of a sum of things is just the slopes of each thing added together!

In this section, we want to think about derivatives of product and quotients of functions. What happens when we differentiate a function that is really just two functions multiplied together?

Activity 2.4.1 Thinking About Derivatives of Products. Let's start with two quick facts:

$$\frac{d}{dx}\left(x^3\right) = 3x^2 \text{ and } \frac{d}{dx}\left(\sin(x)\right) = \cos(x).$$

(a) What is $\frac{d}{dx}\left(x^3 + \sin(x)\right)$? What about $\frac{d}{dx}\left(x^3 - \sin(x)\right)$?

(b) Based on what you just explained, what is a reasonable assumption about what $\frac{d}{dx}\left(x^3 \sin(x)\right)$ might be?

Hint. Does it seem reasonable that we could just multiply the derivatives together, and say that $\frac{d}{dx}\left(x^3 \sin(x)\right)$ was the same thing as

$$\frac{d}{dx}\left(x^3\right) \cdot \frac{d}{dx}\left(\sin(x)\right)?$$

(c) Let's just think about $\frac{d}{dx}\left(x^3\right)$ for a moment. What *is* x^3? Can you write this as a product? Call one of your functions $f(x)$ and the other $g(x)$. You should have that $x^3 = f(x)g(x)$ for whatever you used.

(d) Use your example to explain why, in general, $\frac{d}{dx}\left(f(x)g(x)\right) \neq \frac{d}{dx}\left(f(x)\right) \cdot \frac{d}{dx}\left(g(x)\right)$.

(e) Let's show another way that we know this. Think about $\sin(x)$. We know two things:

$$\sin(x) = (1)(\sin(x)) \text{ and } \frac{d}{dx}\left(\sin(x)\right) = \cos(x).$$

What, though, is $\frac{d}{dx}(1) \cdot \frac{d}{dx}\left(\sin(x)\right)$?

(f) Use all of this to reassure yourself that even though the derivative of a sum of functions is the sum of the derivatives of the functions, we will need to develop a better understanding of how the derivatives of products of functions work.

A thing that I like to think about is this: if $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g'(x)$, then every function's derivative would be 0.

In the example with the $\sin(x)$ function, we noticed that we could write the function as $(1)(\sin(x))$. This is true of every function!

If $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g'(x)$, then we could say that for any function $f(x)$ with a derivative $f'(x)$:

$$\begin{aligned}
\frac{d}{dx}\left(f(x)\right) &= \frac{d}{dx}\left(1 \cdot f(x)\right) \\
&= \frac{d}{dx}(1)\,\frac{d}{dx}\left(f(x)\right) \\
&= 0 \cdot f'(x) \\
&= 0.
\end{aligned}$$

This, obviously, can't be true! We know of *tons* of functions that have non-zero slopes...*most* of them do!

So, we hopefully have some motivation for building a rule to that helps us think about derivatives of products of functions.

2.4.1 The Product Rule

Activity 2.4.2 Building a Product Rule. Let's investigate how we might differentiate the product of two functions:

$$\frac{d}{dx}\left(f(x)g(x)\right).$$

We'll use an area model for multiplication here: we can consider a rectangle where the side lengths are functions $f(x)$ and $g(x)$ that change for different values of x. Maybe x is representative of some type of time component, and the side lengths change size based on time, but it could be anything.

If we want to think about $\frac{d}{dx}\left(f(x)g(x)\right)$, then we're really considering the change in area of the rectangle.

(a) Find the area of the two rectangles. The second rectangle is just one where the input variable for the side length has changed by some amount, leading to a different side length.

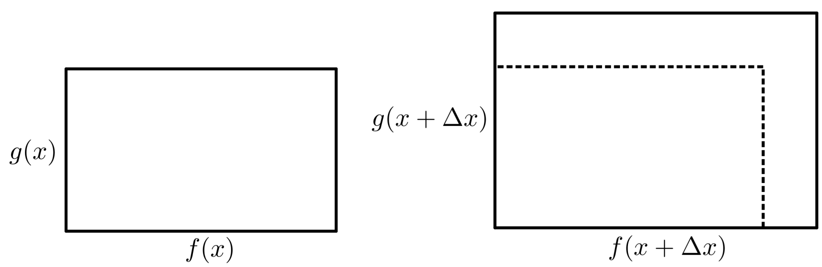


Figure 2.4.1

(b) Write out a way of calculating the difference in these areas.

(c) Let's try to calculate this difference in a second way: by finding the actual area of the region that is new in the second rectangle.

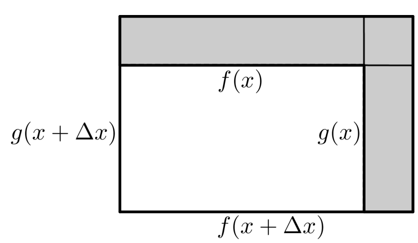


Figure 2.4.2

In order to do this, we've broken the region up into three pieces. Calculate the areas of the three pieces. Use this to fill in the following equation:

$f(x+\Delta x)g(x+\Delta x)-f(x)g(x) =$ ________________.

(d) We do not want to calculate the total change in area: a derivative is a *rate of change*, so in order to think about the derivative we need to divide by the change in input, Δx.

We'll also want to think about this derivative as an *instantaneous* rate of change, meaning we will consider a limit as $\Delta x \to 0$. Fill in the following:

$$\begin{aligned}\frac{d}{dx}\left(f(x)g(x)\right) \lim_{\Delta x \to 0} &\left(\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}\right)\\ &= \lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)\end{aligned}$$

We can split this fraction up into three fractions:

$$\begin{aligned}\frac{d}{dx}\left(f(x)g(x)\right) = &\lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)\\ &+ \lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)\\ &+ \lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)\end{aligned}$$

(e) In any limit with $f(x)$ or $g(x)$ in it, notice that we can factor part out of the limit, since these functions do not rely on Δx, the part that changes in the limit. Factor these out.

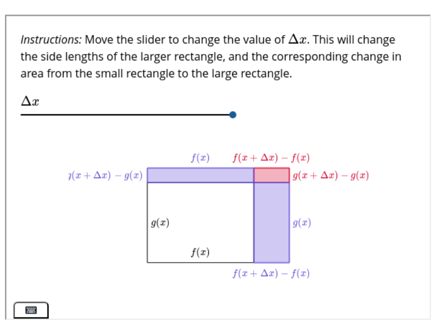
In the third limit, factor out either $\lim_{\Delta x \to 0}\left(f(x+\Delta x) - f(x)\right)$ or $\lim_{\Delta x \to 0}\left(g(x+\Delta x) - g(x)\right)$.

$$\begin{aligned}\frac{d}{dx}\left(f(x)g(x)\right) = f(x) &\lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)\\ + g(x) &\lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)\\ + &\lim_{\Delta x \to 0}\left(\boxed{}\right)\left(\lim_{\Delta x \to 0}\left(\frac{\boxed{}}{\Delta x}\right)\right)\end{aligned}$$

(f) Use Definition 2.1.2 The Derivative Function to re-write these limits. Show that when $\Delta x \to 0$, we get:

$$f(x)g'(x) + g(x)f'(x) + 0.$$

We can investigate this visual a bit more dynamically: see the differences in area as $\Delta x \to 0$. What parts are left, when $\Delta x \to 0$? What areas aren't visible?



Theorem 2.4.3 Product Rule. *If $u(x)$ and $v(x)$ are functions that are differentiable at x and $f(x) = u(x) \cdot v(x)$, then:*

$$\frac{d}{dx}\left(f(x)\right) = u'(x) \cdot v(x) + u(x) \cdot v'(x).$$

For convenience, this is often written as:

$$\frac{d}{dx}(u \cdot v) = u'v + uv' \qquad \textit{or} \qquad \frac{d}{dx}(u \cdot v) = v\left(\frac{du}{dx}\right) + u\left(\frac{dv}{dx}\right).$$

Example 2.4.4 Use the Product Rule to find the following derivatives.

(a) $\frac{d}{dx}\left(x^3 \sin(x)\right)$

Hint. Use $u = x^3$ and $v = \sin(x)$. Now find u' and v', and use:

$$\frac{d}{dx}(uv) = u'v + uv'.$$

(b) $\frac{d}{dx}\left((x^3 + 4x)e^x\right)$

(c) $\frac{d}{dx}\left(\sqrt{x}\cos(x)\right)$

□

2.4.2 What about Dividing?

So we can differentiate a product of functions, and the obvious next question should be about division: if we needed to build a reasonable way of differentiating a product, shouldn't we also need to build a new way of thinking about derivatives of quotients?

A nice thing that we can do is think about division as really just multiplication. For instance, if we want to differentiate $\frac{d}{dx}\left(\frac{\sin(x)}{x^2}\right)$, then we can just think about this quotient as really a product:

$$\frac{d}{dx}\left(\frac{\sin(x)}{x^2}\right) = \frac{d}{dx}\left(\frac{1}{x^2}\left(\sin(x)\right)\right).$$

Now we can just apply product rule!

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{x^2}\left(\sin(x)\right)\right) &= \frac{d}{dx}\left(x^{-2}\sin(x)\right) \\
u = \sin(x) \quad & v = x^{-2} \\
u' = \cos(x) \quad & v' = -2x^{-3} \\
\frac{d}{dx}\left(\sin(x)x^{-2}\right) &= x^{-2}\cos(x) + \left(-2x^{-3}\sin(x)\right) \\
&= \frac{\cos(x)}{x^2} - \frac{2\sin(x)}{x^3}
\end{aligned}$$

This works great! We can *always* think about quotients as just products of reciprocals! But the problem is that we can't always differentiate these reciprocals (for now). We were able to differentiate $\frac{1}{x^2}$ by re-writing this as just a power function (with a negative exponent).

What about this flipped example:

$$\frac{d}{dx}\left(\frac{x^2}{\sin(x)}\right)?$$

In order for us to do the same thing, we need to re-write this as

$$\frac{d}{dx}\left(x^2\left(\sin(x)\right)^{-1}\right)$$

but we don't know how to differentiate $(\sin(x))^{-1}$ (yet).

So let's try to build a general way of differentiating quotients.

Activity 2.4.3 Constructing a Quotient Rule. We're going to start with a function that is a quotient of two other functions:

$$f(x) = \frac{u(x)}{v(x)}.$$

Our goal is that we want to find $f'(x)$, but we're going to shuffle this function around first. We won't calculate this derivative directly!

(a) Start with $f(x) = \dfrac{u(x)}{v(x)}$. Multiply $v(x)$ on both sides to write a definition for $u(x)$.

$u(x) =$ __________

(b) Find $u'(x)$.

(c) Wait: we don't care about $u'(x)$. Right? We care about finding $f'(x)$! Use what you found for $u'(x)$ and solve for $f'(x)$.

$f'(x) =$ __________

(d) This is a strange formula: we have a formula for $f'(x)$ written in terms of $f(x)$! But we said earlier that $f(x) = \dfrac{u(x)}{v(x)}$.

In your formula for $f'(x)$, replace $f(x)$ with $\dfrac{u(x)}{v(x)}$.

$f'(x) =$ __________

This formula is fine, but a little clunky. We'll try to re-write it in some nice ways, but it is a bit more complex than the Product Rule.

Theorem 2.4.5 Quotient Rule. *If $u(x)$ and $v(x)$ are differentiable at x and $f(x) = \dfrac{u(x)}{v(x)}$ then:*

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}.$$

For convenience, this is often written as:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}.$$

Example 2.4.6 Use the Quotient Rule to find the following derivatives.

(a) $\frac{d}{dx}\left(\frac{\sin(x)}{x^2}\right)$

Once you have this derivative, confirm that it is the same as $\frac{\cos(x)}{x^2} - \frac{2\sin(x)}{x^3}$, the way that we found it using the Product Rule.

(b) $\frac{d}{dx}\left(\frac{x^2}{\sin(x)}\right)$

(c) $\frac{d}{dx}\left(\frac{x+4}{x^2+1}\right)$

□

2.4.3 Derivatives of (the Rest of the) Trigonometric Functions

You might remember that of the six main trigonometric functions, we can write four of them in terms of the other two: here are the different trigonometric functions written in terms of sine and cosine functions:

$$\tan(x) = \left(\frac{\sin(x)}{\cos(x)}\right)$$
$$\sec(x) = \left(\frac{1}{\cos(x)}\right)$$
$$\cot(x) = \left(\frac{\cos(x)}{\sin(x)}\right)$$
$$\csc(x) = \left(\frac{1}{\sin(x)}\right)$$

Example 2.4.7 Find the derivatives of the remaining trigonometric functions.

(a) $\frac{d}{dx}(\tan(x))$

Hint. Write $\frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$ and use the Quotient Rule.

(b) $\frac{d}{dx}(\sec(x))$

Hint. Write $\frac{d}{dx}(\sec(x)) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right)$ and use the Quotient Rule.

(c) $\frac{d}{dx}(\cot(x))$

Hint. Write $\frac{d}{dx}(\cot(x)) = \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right)$ and use the Quotient Rule.

(d) $\frac{d}{dx}(\csc(x))$

Hint. Write $\frac{d}{dx}(\csc(x)) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right)$ and use the Quotient Rule.

□

2.5 The Chain Rule

We've been building up some intuition and rules to help us think about differentiating different functions and combinations of functions. We can find derivatives of scaled functions, sums of functions, differences of functions, products of functions, and also quotients of functions.

In this section, we'll look at our last operation between functions: composition.

2.5.1 Composition and Decomposition

An important part of finding derivatives of products and quotients is identifying the component functions that are being multiplied/divided (often labeled $u(x)$ or just u and $v(x)$ or just v). From there, we find the derivatives of each of the component functions, and then use the formula from the Product Rule or Quotient Rule to put the pieces together.

Thinking about derivatives of composed functions will be the same: we'll need to identify what functions are being composed inside of other functions, and use those pieces in some formulaic way to represent the derivative. On that note, let's remind ourselves and practice working with composition (and decomposition) of functions.

Activity 2.5.1 Composition (and Decomposition) Pictionary. This activity will involve a second group, or at least a partner. We'll go through the first part of this activity, and then connect with a second group/person to finish the second part.

(a) Build two functions, calling them $f(x)$ and $g(x)$. Pick whatever kinds of functions you'd like, but this activity will work best if these functions are in a kind of sweet-spot between "simple" and "complicated," but don't overthink this.

(b) Compose $g(x)$ inside of $f(x)$ to create $(f \circ g)(x)$, which we can also write as $f(g(x))$.

(c) Write your composed $f(g(x))$ function on a separate sheet of paper. Do not leave any indication of what your chosen $f(x)$ and $g(x)$ are. Just write your composed function by itself.

Now, pass this composed $f(g(x))$ to your partner/a second group.

(d) You should have received a new function from some other person/group. It is different than yours, but also labeled $f(g(x))$ (with different choices of $f(x)$ and $g(x)$).

Identify a possibility for $f(x)$, the outside function in this composition, as well as a possibility for $g(x)$, the inside function in this composition. You can check your answer by composing these!

(e) Write a different pair of possibilities for $f(x)$ and $g(x)$ that will still give you the same composed function, $f(g(x))$.

(f) Check with your partner/the second group: did you identify the pair of functions that they originally used?

Did whoever you passed your composed function to correctly identify your functions?

A big thing to notice here is that when we pick the pieces of functions that we think were composed inside of each other, there's not a single obvious answer. This is pretty different compared to, say, using the Quotient Rule. In these quotients, we have a natural division (literally) between the pieces. Here, it's much more subjective for us when we decide to label an "inside" function and an "outside" function.

We will build up our intuition to find a good balance for how we pick these.

2.5.2 The Chain Rule, Intuitively

Before we build the Chain Rule for differentiating composed functions, we should talk about some notation. Earlier (in Subsection 2.2.5) we talked about the derivative notation, $\frac{dy}{dx}$. One of the things we mentioned is that while we know that the derivative is an instantaneous rate of change, this notation is helpful to tell us *what* is changing with regard to *what*.

In $\frac{dy}{dx}$, we are calculating how much the y-variable changes when x increases. If we talked about $\frac{df}{dt}$, then we are discussing how much f changes for an increase in t, whatever these variables represent.

Activity 2.5.2 Gears and Chains. Let's think about some gears. We've got three gears, all different sizes. But the gears are linked together, and a small motor works to spin one of the gears. Since the gears are linked, when one gear spins, they all do. But since they are different sizes, they complete a different number of revolutions: the smaller ones spin more times than the larger ones, since they have a smaller circumference.

For our purpose, let's say that Gear A is being driven by the motor.

(a) Let's try to quantify how much "faster" Gear B is spinning compared to Gear A. How many revolutions does Gear B complete in the time it takes Gear A to complete one revolution?

(b) Now quantify the speed of Gear C compared to its neighbor, Gear B. How many revolutions does Gear C complete in the time it takes Gear B to complete one revolution?

(c) Use the above relative "speeds" to compare Gear C and Gear A: how many revolutions does Gear C complete in the time it takes Gear A to complete one revolution?

More importantly, how do we find this?

(d) Now let's translate this into some derivative notation: we've really been finding rates at which one thing changes (the speed of the gear spinning) relative to another's.

Call the speed of Gear B compared to Gear A: $\frac{dB}{dA}$. Now call the speed of Gear C compared to Gear B: $\frac{dC}{dB}$. Come up with a formula to find $\frac{dC}{dA}$.

So what we need to do now is to somehow translate this intuitive idea of multiplying rates of change to build a strategy for thinking about derivatives of composed functions.

We can think of these linked gears as functions: Gear C changes based on what is happening with Gear B, which changes based on Gear A. We can translate Gear A to be an input variable, like x. Then Gear B is a function based on that: we can call it $g(x)$. Then Gear C is a function that takes in the position of Gear B (the function $g(x)$), and so we can think of it as $f(g(x))$.

To build the derivative rule for composite functions, we need to find how the ”outside” function changes as the ”inside” function changes ($\frac{dC}{dB}$ in this case) and multiply that by how the ”inside” function changes as the input variable changes ($\frac{dB}{dA}$ here).

Theorem 2.5.1 The Chain Rule. *For the composite function $y = f(g(x))$, if we define $u = g(x)$ and $y = f(u)$, then, as long as both f and g are differentiable at u and x respectively:*

$$\frac{d}{dx}\left(f(g(x))\right) = \frac{d}{du}\left(f(u)\right) \cdot \frac{d}{dx}\left(g(x)\right).$$

Alternatively, this can be written as:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \textit{or} \qquad \frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x).$$

2.5.3 Doing is Different than Knowing

It is lovely to know that the Chain Rule is really just linking the two rates of change together to connect a function with an input variable through a middle processing function. That’s great!

But doing the Chain Rule is different than just knowing it, so let’s walk through a first example. Let’s find the following derivative:

$$\frac{d}{dx}\left(\sin(x^2)\right)$$

We’ll call the ”inside” function $u = x^2$, so we can really write the whole function (normally we’re calling this y) as $y = \sin(u)$.

$$\begin{aligned}\frac{d}{dx}\left(\underbrace{\sin(\overbrace{x^2}^{u})}_{y}\right) &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= \frac{d}{du}\left(\sin(u)\right) \cdot \frac{d}{dx}\left(x^2\right)\end{aligned}$$

What we can notice, here, is that $\sin(u)$ is just a function of some variable u, and we want to find $\frac{dy}{du}$, the rate at which $y = \sin(u)$ changes with regard to its input variable. This might feel a bit strange, since u isn’t just an input variable: it *means* something, since we have that $u = x^2$. This is fine! The extra $\frac{du}{dx}$ that we multiply will take care of linking this derivative to the input variable x.

$$\begin{aligned}\frac{d}{dx}\left(\underbrace{\sin(\overbrace{x^2}^{u})}_{y}\right) &= \frac{d}{du}\left(\sin(u)\right) \cdot \frac{d}{dx}\left(x^2\right)\\ &= \cos(u) \cdot 2x\\ &= \cos(x^2) \cdot 2x\\ &= 2x\cos(x^2)\end{aligned}$$

After we finished differentiating $\frac{d}{du}\left(\sin(u)\right)$, you’ll notice that we used the fact that $u = x^2$ to write our combination of derivatives (the derivative of the ”outside” function and the derivative of the ”inside” function) in terms of the same input variable again.

The last line, rewriting $\cos(x^2) \cdot 2x$ as $2x\cos(x^2)$, is just for aesthetics.

Now you're ready to try some more examples! In each, focus on identifying a natural selection for the "inside" function, u.

Example 2.5.2 Use the Chain Rule to differentiate the following:

(a) $\frac{d}{dx}\left(\sqrt{x^2+4}\right)$

Hint. Notice that x^2+4 is composed under the square root. Use $u = x^2+4$.

(b) $\frac{d}{dx}\left(e^{\tan(x)}\right)$

Hint. Try letting $u = \tan(x)$, since it's composed inside the exponent of the exponential function.

(c) $\frac{d}{dx}\left(\sin^5(x)\right)$

Hint. You could think about this as $\frac{d}{dx}\left(\sin(x)\sin(x)\sin(x)\sin(x)\sin(x)\right)$ and try to use a very annoying product rule, but it might be easier to think about this as $\frac{d}{dx}\left((\sin(x))^5\right)$.

□

Chapter 3

Implicit Differentiation

3.1 Implicit Differentiation

3.1.1 Using a Derivative as an Operator

3.2 Derivatives of Inverse Functions

Text of section.

3.3 Logarithmic Differentiation

Text of section.

3.4 Related Rates

Text of section.

Chapter 4

Applications of Derivatives

4.1 Mean Value Theorem

Text

4.1.1 Slopes

Text

4.1.2 The Mean Value Theorem

Theorem 4.1.1 Mean Value Theorem. *For a function $f(x)$ that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is some value $x = c$ with $a < c < b$ where:*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

4.1.3 More Results due to the Mean Value Theorem

Theorem 4.1.2 *For two functions f and g, both differentiable on (a, b), if $f'(x) = g'(x)$ for all x-values on (a, b), then we know that $f(x) = g(x) + C$ for some real number constant C. That is, the only differences in f and g are due to a difference in the constant term.*

4.2 Increasing and Decreasing Functions

Text of section.

4.3 Concavity

Text of section.

4.4 Interpreting the First and Second Derivatives

Text of section.

4.5 Global Maximums and Minimums

Text of section.

4.6 Optimization

Text of section.

4.7 Linear Approximations

Text of section.

4.8 Newton's Method for Approximating Zeros

Text of section.

4.9 L'Hopital's Rule

Text of section.

Theorem 4.9.1 L'Hopital's Rule. *If $f(x)$ and $g(x)$ are functions and a is some real number with f and g both being differentiable at a and $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Similarly, this holds if $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$.

If f and g are both differentiable as $x \to \infty$ and either:

- $\lim_{x\to\infty} f(x) = 0$ *and* $\lim_{x\to\infty} g(x) = 0$
- $\lim_{x\to\infty} f(x) = \pm\infty$ *and* $\lim_{x\to\infty} g(x) = \pm\infty$

then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

This is also true as $x \to -\infty$.

Chapter 5

Antiderivatives and Integrals

5.1 Antiderivatives and Indefinite Integrals

5.2 Riemann Sums and Area Approximations

Text of section.

Definition 5.2.1 Riemann Sum. For a closed interval $[a, b]$ with a partition $P = \{x_0, x_1, ..., x_n\}$ with $a = x_0 < x_1 < ... < x_n = b$, consider some x_k^*, any x-value in the interval $[x_{k-1}, x_k]$ and Δx_k, the length of the interval $[x_{k-1}, x_k]$. If f is a function that is defined on the interval $[a, b]$, then we call the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + ... + f(x_n^*)\Delta x_n$$

a **Riemann Sum** for f on $[a, b]$. ♢

In practice, we typically choose a *Regular Partition*, where each subinterval $[x_{k-1}, x_k]$ is equally-wide, and so $\Delta x_k = \frac{b-a}{n}$ for every $k = 1, 2, ..., n$. We then normally write our Riemann sum as

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + ... + f(x_n^*)\Delta x.$$

5.3 Definite Integral

The Definite Integral

5.4 The Fundamental Theorem of Calculus

Theorem 5.4.1 Fundamental Theorem of Calculus (Part 1). *For a function f that is continuous on an interval $[a, b]$, and a function $A(x) = \int_{t=a}^{t=x} f(t)\, dt$ defined for x-values in $[a, b]$, then $A'(x) = f(x)$. That is:*

$$\frac{d}{dx}\left(\int_{t=a}^{t=x} f(t)\, dt\right) = f(x).$$

Theorem 5.4.2 Fundamental Theorem of Calculus (Part 2). *For a function $f(x)$ continuous on the closed interval $[a,b]$ and some $F(x)$, an antiderivative of $f(x)$ ($F'(x) = f(x)$ on $[a,b]$), then*

$$\int_{x=a}^{x=b} f(x)\,dx = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a).$$

5.5 More Definite Integrals

Text of section.

Chapter 6

Applications of Integrals

Text here.

6.1 Integrals as Net Change

We have some rudimentary ideas of what an integral is, but we want to challenge and expand those ideas by examining the object at the root of the definition of the definite integral: a Riemann sum.

6.1.1 Estimating Movement

Here is some text leading to our first activity.

Activity 6.1.1 Estimating Movement. We're observing an object traveling back and forth in a straight line. Throughout a 5 minute interval, we get the following information about the velocity (in feet/second) of the object.

Table 6.1.1 Velocity of an Object

t	$v(t)$
0	0
30	2
60	4.25
90	5.75
120	3.5
150	0.75
180	-1.25
210	-3.5
240	-2.75
270	-0.5
300	-0.25

(a) Describe the motion of the object in general.

Hint. How do we interpret the different values of velocity? How do we interpret the sign of velocity? What about how velocity changes from one of the 30-second time points to the next?

(b) When was the acceleration of the object the greatest? When was it the least?

Hint. You can decide how to interpret the "least" acceleration: it is either where the acceleration is closes to 0, or it is the most negative value of the acceleration. These are interpreted differently, but it's a bit ambiguous what we might mean when we say "least acceleration."

(c) Estimate the total displacement of the object over the 5 minute interval. What is the overall change in position from the start to the end?

Hint. How do we use velocity and some time interval to estimate the distance traveled? How do we estimate/assume the velocity on each 30-second time interval?

(d) Is this different than the total distance that the object traveled over the 5 minute interval? Why or why not?

Hint. How do we think about (or ignore) the direction of the object? Why is this important here?

(e) If we know the initial position of the object, how could we find the position of the object at some time, t, where t is a multiple of 30 between 0 and 300?

Hint. Can we limit the time intervals that we use to calculate the object's displacement? How do we use displacement and a starting point to find an ending point?

So what are the big ideas in this short activity? There are a lot, and many of them are already things we know, at least to some level. So we are really focusing on adding depth to our understanding of these big ideas. Let's list them in the order that they showed up in this activity:

1. We interpret the velocity as the derivative of the position of the object. So when we interpret the value of the velocity of the object (large vs small, positive vs negative, etc.) we are interpreting these through the lens of a rate of change.

2. Acceleration is the derivative of the velocity function. While we don't have the full picture of the velocity function at any value of t, we still were interested in the rates at which velocity changes with regard to time.

3. We can estimate the total *displacement* of the object by predicting how far it traveled in eahc 30-second time interval. We might pick the starting velocity for each 30-second interval and multiply that by 30 seconds. We could alternatively pick the ending velocity of each 30-second interval. Then we can add all of these products of velocity and time together to approximate a total change in position! Doesn't this feel like a Riemann sum?

4. When we calculate displacement, the negative velocities get multiplied out to get negative changes in position for the object -- that's because a negative velocity means that the object is moving backwards. If we wanted to calculate the distance traveled, then we need to not account for negative velocities. We can just disregard the sign of the velocity on each time interval and repeat the process above. So, another Riemann sum then?

5. In order to forecast some position at time t, we just need to start with the initial position, and then calculate (or approximate) the displacement from $t = 0$ to whatever time $t \leq 300$ we care about, and then add the displacement to the intitial position.

Ok, now let's formalize those results!

6.1.2 Position, Velocity, and Acceleration

We know that the velocity of an object is really a rate of change of the position of that object with regard to time. Similarly, the acceleration of an object is the rate of change of the velocity of the object with regard to time. So we're really thinking about derivatives!

Definition 6.1.2 For an object moving along a straight line, if $s(t)$ represents the **position** of that object at time t, then the **velocity** of theobject at time t is $v(t) = s'(t)$ and the **acceleration** of the object at time t is $a(t) = v'(t) = s''(t)$. ♢

Once we establish this relationship, we can answer questions about movement of an object using the same interpretations of derivatives that we practiced in Chapter 3 of this text.

Activity 6.1.2 A Friendly Jogger. Consider a jogger running along a straight-line path, where their velocity at t hours is $v(t) = 2t^2 - 8t + 6$, and velocity is measured in miles per hour. We begin observing this jogger at $t = 0$ and observe them over a course of 3 hours.

(a) When is the jogger's acceleration equal to 0 mi/hr^2?

Hint. Solve $a(t) = v'(t) = 0$.

(b) Does this time represent a maximum or minimum velocity for the jogger?

Hint. You can use the First Derivative Test or the Second Derivative Test here!

(c) When is the jogger's velocity equal to 0 mi/hr?

(d) Describe the motion of the jogger, including information about the direction that they travel and their top speeds.

6.1.3 Displacement, Distance, and Speed

Let's revisit Activity 6.1.1. When we approximated the displacement of the object, we built a Riemann sum:

$$\sum_{k=1}^{10} v(t_k^*)\Delta t$$

We chose our t_k^* as either the time at the beginning of each 30-second interval or the time atr the end of the 30-second interval, but that was only because of the limited information that we had about different values of $v(t)$. If we had information about the $v(t)$ function at any values of t $(0 \le t \le 300)$, then we could pick *any* time in each 30-second time interval for our Riemann sum! We might note, though, that if we did have this kind of information about the velocity at any time in the 5-minute interval, then we would also build a more precise approximation by subdividing the time interval into smaller/shorter pieces. So maybe the Riemann sum $\sum_{k=1}^{100} v(t_k^*)\Delta t$ (where we are dividing up the 5 minute interval into 100 3-second intervals) would do a better job! But why stop there? If we have the definition of the velocity function, and so we can truly obtain the velocity of the object at *any* time in the 5 minute interval, then we can use the definition of the definite integral as the limit of a Riemann sum:

$$\lim_{n\to\infty}\sum_{k=1}^{n} v(t_k^*)\Delta t = \int_{t=0}^{t=300} v(t)\ dt$$

This should work out well with our first understanding of displacement: the displacement of an object is just the difference in position from the starting time to the ending time. So we could say that if $s(t)$ is the position function, then we might expect to represent displacement from $t = a$ to $t = b$ as $s(b) - s(a)$. But isn't this just the Fundamental Theorem of Calculus, since $s'(t) = v(t)$?

Definition 6.1.3 If an object is moving along a straight line with velocity $v(t)$ and position $s(t)$, then the **displacement** of the object from time $t = a$ to $t = b$ is

$$\int_{t=a}^{t=b} v(t)\ dt = s(b) - s(a)$$

◊

Let's keep revisiting the same activity. We also noticed that when we looked at the *distance* compared to the displacement, the only difference was that we were integrating the absolute value of the velocity function, since we didn't care about the sign of the velocity (the direction that the object was traveling) on each interval.

Definition 6.1.4 If an object is moving along a straight line with velocity $v(t)$, then the **distance** traveled by the object from time $t = a$ to $t = b$ is:

$$\int_{t=a}^{t=b} |v(t)|\ dt$$

Here, we call $|v(t)|$ the **speed** of the object (instead of the velocity). ◊

We should note that we don't have any quick and easy ways of dealing with the integral of the absolute value of a function.

$$v(t)| = \begin{cases} -v(t) & \text{when } v(t) < 0 \\ v(t) & \text{when } v(t) \geq 0 \end{cases}$$

So, in order for us to integrate $|v(t)|$, we need to think about where the velocity passes through 0, so that we can see where it might change from positive to negative.

Activity 6.1.3 Tracking our Jogger. Let's revisit our jogger from Activity 6.1.2.

(a) Calculate the total displacement of the jogger from $t = 0$ to $t = 3$.

Hint. Set up and evaluate a definite integral here, using the velocity function.

(b) Think back to our descroption of the jogger's movement: when is this jogger moving backwards? Split up the time interval from $t = 0$ (the start of their run) to $t = c$ (where c is the time that the jogger changed direction) to $t = 3$. Calculate the displacements on each of these two intervals.

(c) Calculate the total distance that the jogger traveled in their 3 hour run.

Hint. Remember that we're really calculating:

$$\left|\int_{t=0}^{t=c} v(t)\, dt\right| + \left|\int_{t=c}^{t=3} v(t)\, dt\right|$$

6.1.4 Finding the Future Value of a Function

We can again think back to Activity 6.1.1 and build our last result of this section. Remember when we were looking to predict the location of our object at different times: we said it was reasonable to start at our initial position, and then add the displacement of the object from that initial time up to the time that we were interested in. So, to estimate the object's position after 150 seconds, we would calculate:

$$s(0) + \int_{t=0}^{t=150} v(t)\, dt.$$

But we said we could do this to estimate the object's position at any value for time, t.

Definition 6.1.5 Future Position of an Object. For some object moving along a straight line with velocity $v(t)$ and an intitial position of $s(a)$, the **future position of the object** at some time t (with $t \geq a$) is:

$$\underbrace{s(t)}_{\substack{\text{future}\\\text{position}}} = \underbrace{s(0)}_{\substack{\text{initial}\\\text{position}}} + \underbrace{\int_{x=a}^{x=t} v(x)\, dx}_{\substack{\text{displacement from}\\ a \text{ to } t}}$$

Note that we change the variable in the velocity function while we integrate: since we want our position function to be in terms of t, the ending time point that we calculate the displacement up to, we need to choose a different variable to write velocity in terms of. Mechanically, there is no difference, since we're just swapping out the variables and naming them x. ◊

We can note that this relationship between velocity and position can exist in many other context: any pair of functions that are derivatives/antiderivatives of each other can have this relationship!

Definition 6.1.6 Net Change and Future Value. Suppose the value $F(t)$ changes over time at a known rate $F'(t)$. Then the **net change** in F between $t = a$ and $t = b$ is:

$$F(b) - F(a) = \int_{t=a}^{t=b} F'(t)\, dt.$$

Similarly, given the initial value $F(a)$, the **future value** of F at time $t \geq a$ is:

$$F(t) = F(a) + \int_{x=a}^{x=t} F'(x)\, dx$$

◊

6.2 Area Between Curves

Let's remember Riemann Sums.

6.2.1 Remembering Riemann Sums

Text here.

Activity 6.2.1 Remembering Riemann Sums. Let's start with the function $f(x)$ on the interval $[a, b]$ with $f(x) > 0$ on the interval. We will construct a Riemman sum to approximate the area under the curve on this interval, and then build that into the integral formula.

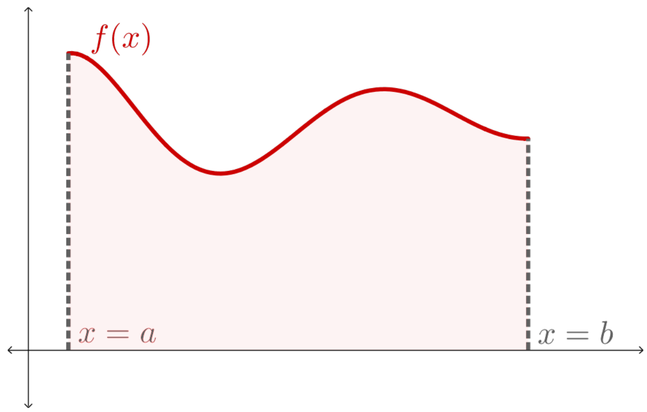


Figure 6.2.1

(a) Divide the interval $[a, b]$ into 4 equally-sized subintervals.

(b) Pick an x_k^* for $k = 1, 2, 3, 4$, one for each subinterval. Then, plot the points $(x_1^*, f(x_1^*))$, $(x_2^*, f(x_2^*))$, $(x_3^*, f(x_3^*))$, and $(x_4^*, f(x_4^*))$.

Hint. These points are just general ones, and you don't have to come up with actual numbers for the x-values or the corresponding y-values. Instead, just draw them in on the curve, somewhere in each of the subintervals.

(c) Use these 4 points to draw 4 rectangles. What are the dimensions of these rectangles (the height and width)?

Hint. You won't have any numbers to calculate here, really: instead, see if you can calculate the widths by thinking about the total width of your interval. Then calculate the heights by thinking about the points you created.

(d) Find the area of each rectangle by multiplying the heights and widths for each rectangle.

(e) Add up the areas to construct a Riemman sum. Is this sum very accurate? Why or why not?

Hint. Try to think about the accuracy of your area approximation by looking at it visually. Are there any places where your approximation looks far away from the actual area we're thinking about?

(f) Now we will generalize a little more. Let's say we divide this up into n equally-sized pieces (instead of 4). Instead of trying to pick an x_k^* for the unknown number of subintervals (since we don't have a value for n yet), let's just focus on one of these: the arbitrary kth subinterval.

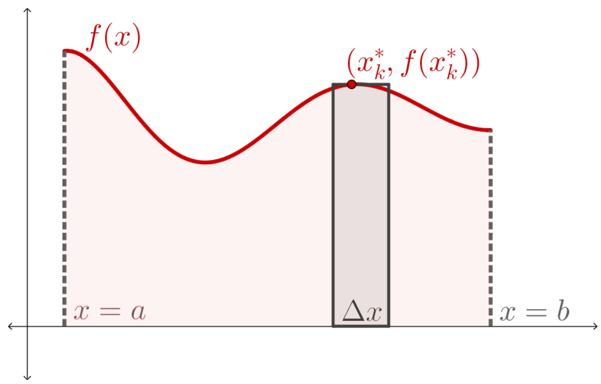


Figure 6.2.2

What are the dimensions of this kth rectangle?

(g) Find A_k, the area of this kth rectangle.

(h) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

Hint. You might want to use summation notation, starting with $\sum_{k=1}^{n}$

(i) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area under the curve $f(x)$ from $x = a$ to $x = b$.

This activity hopefully reminds of the definition of a Riemann Sum (Definition 5.2.1) from earlier in this text (Section 5.2).

6.2.2 Building an Integral Formula for the Area Between Curves

Activity 6.2.2 Area Between Curves. Let's start with our same function $f(x)$ on the same interval $[a, b]$m but also add the function $g(x)$ on the same interval, with $f(x) > g(x) > 0$ on the interval. We will construct a Riemann sum to approximate the area between these two curves on this interval, and then build that into the integral formula.

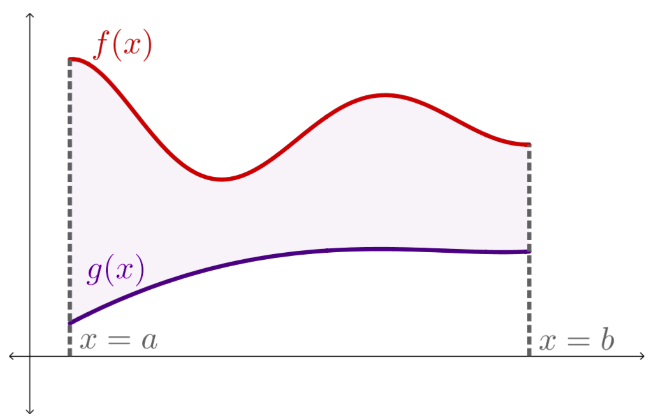


Figure 6.2.3

(a) Divide the interval $[a, b]$ into 4 equally-sized subintervals.

(b) Pick an x_k^* for $k = 1, 2, 3, 4$, one for each subinterval. Plot the points $(x_1^*, f(x_1^*))$, $(x_2^*, f(x_2^*))$, $(x_3^*, f(x_3^*))$, and $(x_4^*, f(x_4^*))$. Then plot the corresponding points on the g function: $(x_1^*, g(x_1^*))$, $(x_2^*, g(x_2^*))$, $(x_3^*, g(x_3^*))$, and $(x_4^*, g(x_4^*))$.

(c) Use these 8 points to draw 4 rectangles, with the points on the f function defining the tops of the rectangles and the points on the g function defining the bottoms of the rectangles. What are the dimensions of these rectangles (the height and width)?

(d) Find the area of each rectangle by multiplying the heights and widths for each rectangle.

(e) Add up the areas to construct a Riemman sum.

(f) Now we will generalize a little more. Let's say we divide this up into n equally-sized pieces (instead of 4). Instead of trying to pick an x_k^* for the unknown number of subintervals (since we don't have a value for n yet), let's just focus on one of these: the arbitrary kth subinterval.

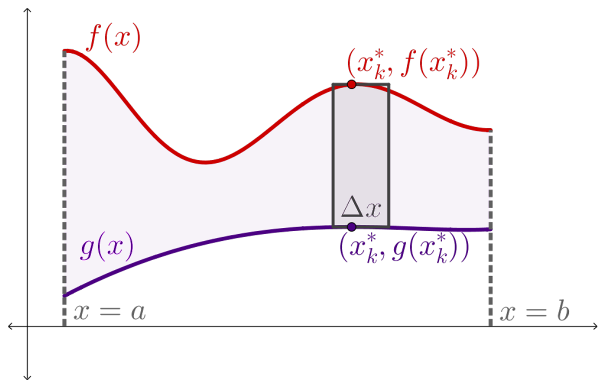


Figure 6.2.4

What are the dimensions of this kth rectangle?

(g) Find A_k, the area of this kth rectangle.

(h) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

Hint. You might want to use summation notation, starting with $\sum_{k=1}^{n}$

(i) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$.

Definition 6.2.5 Area Between Curves. If $f(x)$ and $g(x)$ are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$, then the **area bounded between the curves** $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ is

$$A = \int_{x=a}^{x=b} (f(x) - g(x))\ dx.$$

◊

Example here.

6.2.3 Changing Perspective

Text here.

Activity 6.2.3 Trying for a Single Integral. Let's consider the same setup as earlier: the region bounded between two curves, $y = x + 6$ and $y = x^3$, as well as the x-axis (the line $y = 0$). We'll need to name these functions, so let's call them $f(x) = x^3$ and $g(x) = x + 6$. But this time, we'll approach the region a bit differently: we're going to try to find the area of the region using only a single integral.

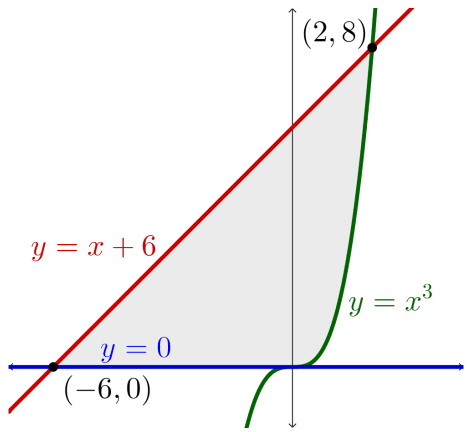


Figure 6.2.6

(a) The range of y-values in this region span from $y = 0$ to $y = 8$. Divide this interval evenly into 4 equally sized-subintervals. What is the height of each subinterval? We'll call this Δy.

Hint. $\Delta y = \dfrac{8-0}{4}$

(b) Pick a y-value from each sub-interval. You can call these y_1^*, y_2^*, y_3^*, and y_4^*.

(c) Find the corresponding x-values on the $f(x)$ function for each of the y-values you selected. These will be $f^{-1}(y_1^*)$, $f^{-1}(y_2^*)$, $f^{-1}(y_3^*)$, and $f^{-1}(y_4^*)$.

Hint. You're really just putting your y-values into the equation $y = x + 6$ and solving for x. Or you can solve for $f^{-1}(y)$ in general, by solving for x while leaving y as a variable.

(d) Do the same thing for the g function. Now you have 8 points that you can plot: $\left(f^{-1}(y_1^*), y_1^*\right)$, $\left(f^{-1}(y_2^*), y_2^*\right)$, $\left(f^{-1}(y_3^*), y_3^*\right)$, and $\left(f^{-1}(y_4^*), y_4^*\right)$ as well as $\left(g^{-1}(y_1^*), y_1^*\right)$, $\left(g^{-1}(y_2^*), y_2^*\right)$, $\left(g^{-1}(y_3^*), y_3^*\right)$, and $\left(g^{-1}(y_4^*), y_4^*\right)$. Plot them.

(e) Use these points to draw 4 rectangles with points on f and g determining the left and right ends of the rectangle. What are the dimensions of these rectangles (height and width)?

(f) Find the area of each rectangle by multiplying the height and widths for each rectangle.

(g) Add up the areas to construct a Riemann sum.

(h) Again, we'll generalize this and think about the kth rectangle, pictured below.

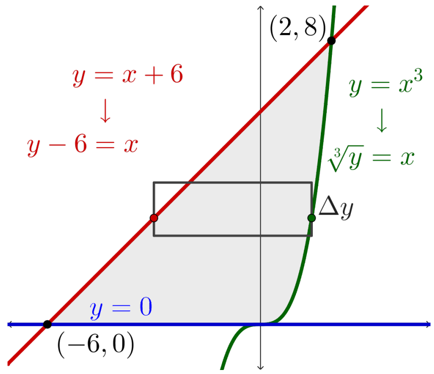


Figure 6.2.7

Which variable defines the location of the kth rectangle, here? That is, if you were to describe *where* in this graph the kth rectangle is laying, would you describe it with an x or y variable? This will act as our general input variable for the integral we're ending with.

(i) What are the dimensions of the kth rectangle?

(j) Find A_k, the area of this kth rectangle.

(k) Add up the areas of A_k for $k = 1, 2, 3, ..., n$ to approximate the total area, A

Hint. You might want to use summation notation, starting with $\sum_{k=1}^{n}$

(l) Apply a limit as $n \to \infty$ to this Riemann sum in order to construct the integral formula for the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$.

(m) Now that you have an integral, evaluate it! Find the area of this region to compare with the work we did previously, where we used multiple integrals to measure the size of this same region.

We can re-write our definition of the area between curves (Definition 6.2.5) to account for this change in perspective, by thinking about these same functions in terms of y.

Definition 6.2.8 Area Between Curves (in terms of y). If $f(y)$ and $g(y)$ are continuous functions with $f(y) \geq g(y)$ on the interval of y-values $[c, d]$, then the **area bounded between the curves** $x = f(y)$ and $x = g(y)$ from $y = c$ to $y = d$ is

$$A = \int_{y=c}^{y=d} (f(y) - g(y))\ dy.$$

◊

6.2.4 Practice Problems

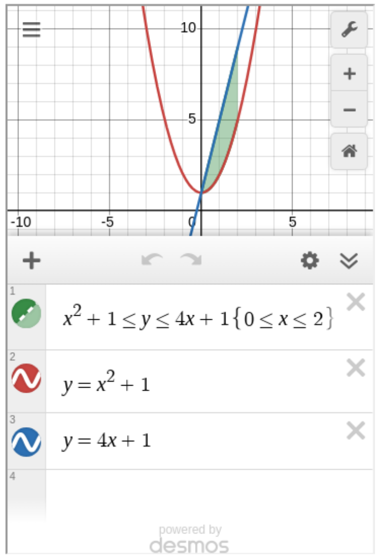
1. Explain how we use the "slice and sum" method to build an integral formula for the area bounded between curves. Give some details, enough to make sure you understand how the Riemann sums are constructed and how they turn into our integral formula.

2. What are some changes/considerations that we need to make when we decide to set up our integral in terms of y instead of x?

3. Set up (and practice evaluating) an integral expression representing the area of each of the regions described below.

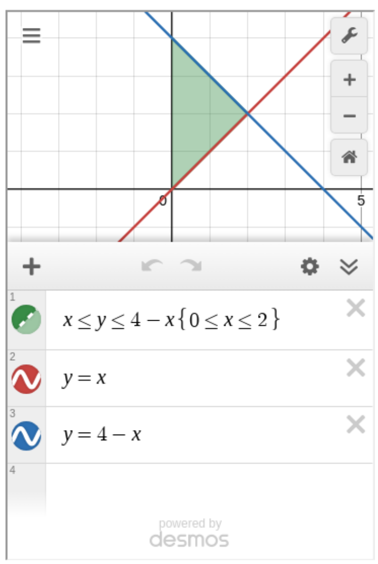
(a) The region bounded by the curves $y = x^2 + 1$ and $y = 4x + 1$ between $x = 0$ and $x = 2$.

Hint.



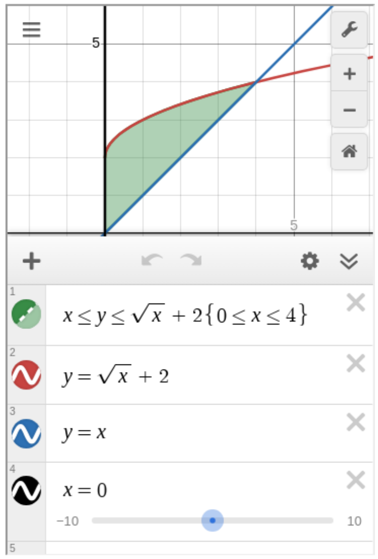
(b) The region bounded by the curves $y = x$ and $y = 4 - x$ between $x = 0$ and $x = 2$

Hint.



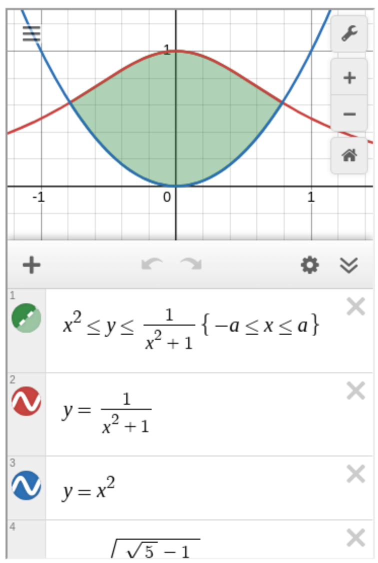
(c) The region bounded by the curves $y = \sqrt{x} + 2$ and $y = x$ and the line $x = 0$.

Hint.



(d) The region bounded by the curves $y = \dfrac{2}{x^2+1}$ and $y = x^2$.

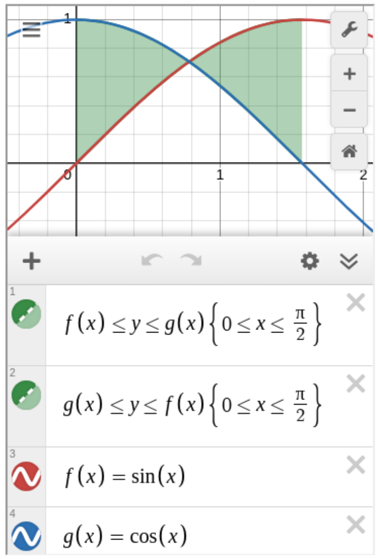
Hint.



4. Set up and evaluate an integral representing the area of each of the regions described below. Explain whether you chose to integrate with respect to x or y, and why you made that choice.

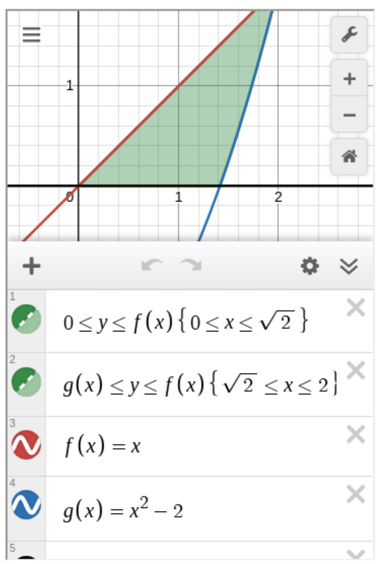
(a) The region bounded by the curves $y = \sin(x)$ and $y = \cos(x)$ and the line $y = 0$ between $x = 0$ and $x = \frac{\pi}{2}$.

Hint.



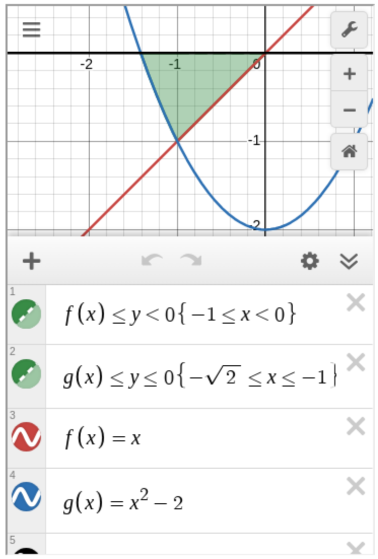
(b) The region bounded by the curves $y = x$ and $y = x^2 - 2$ and the line $y = 0$ in the first quadrant.

Hint.



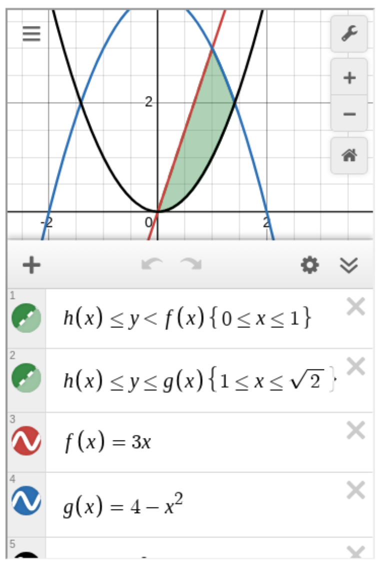
(c) The region bounded by the curves $y = x$ and $y = x^2 - 2$ and the line $y = 0$ in the third quadrant.

Hint.



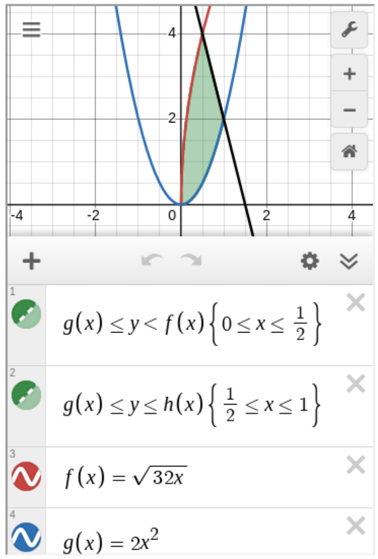
(d) The region bounded by the curves $y = 3x$, $y = 4 - x^2$, and $y = x^2$ in the first quadrant.

Hint.



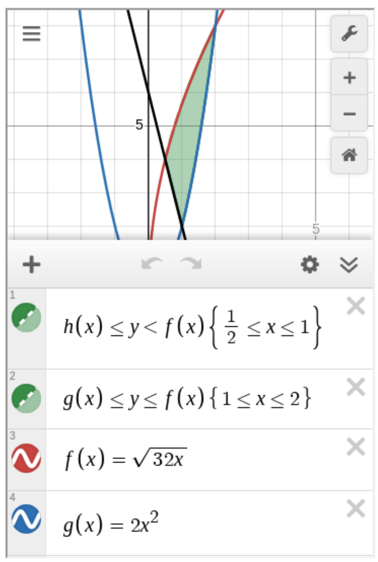
(e) The region bounded by the curves $y = \sqrt{32x}$, $y = 2x^2$, and $y = -4x + 6$ in the first quadrant.

Hint.



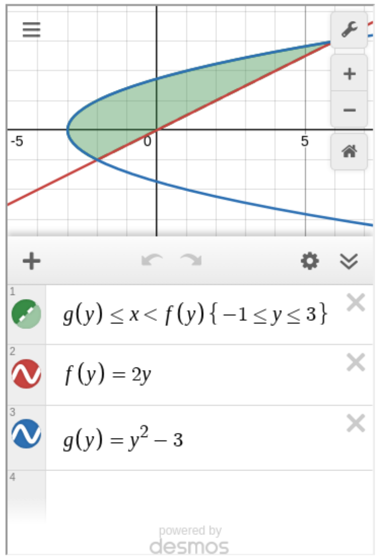
(f) The *other* region bounded by the curves $y = \sqrt{32x}$, $y = 2x^2$, and $y = -4x + 6$ in the first quadrant.

Hint.



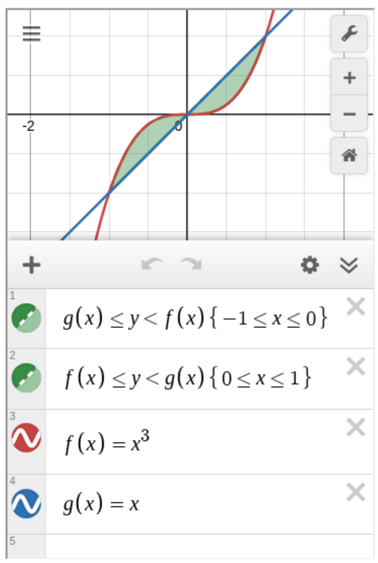
(g) The region bounded by the curves $x = 2y$ and $x = y^2 - 3$.

Hint.



(h) The region(s) bounded by the curves $y = x^3$ and $y = x$.

Hint.



6.3 Volumes of Solids of Revolution

6.3.1 From Area To Volume

Text.

Definition 6.3.1 Volume by Disks/Washers. If f and g are continuous functions with $f(x) \geq g(x) \geq 0$ on the interval $[a, b]$, then the volume of the solid formed by revolving the region bounded between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ around the x-axis is:

$$V = \pi \int_{x=a}^{x=b} \left((f(x))^2 - (g(x))^2\right)\, dx.$$

This is called the **Washer Method**. Note that if $g(x) = 0$, then the resulting volume is:

$$V = \pi \int_{x=a}^{x=b} (f(x))^2 \; dx.$$

This is called the **Disk Method**. ◊

Activity 6.3.1 Volumes by Disks/Washers. Consider the region bounded between the curves $y = 4 + 2x - x^2$ and $y = \frac{4}{x+1}$. Will will create a 3-dimensional solid by revolving this region around the x-axis.

(a) Visualize the solid you'll create when you revolve this region around the x-axis.

(b) Draw a single rectangle in your region, standing perpendicular to the x-axis.

(c) Let's use this rectangle to visualize the kth slice of this 3-dimensional solid. What does the "face" of it look like?

(d) Find the area of the face of the kth slice.

Hint. Note that this is a 2-dimensional shape, and we're just finding the area of it.

(e) Set up, and evaluate, the integral representing the volume of the solid.

Activity 6.3.2 Another Volume. Now lets consider another region: this time, the one bounded between the curves $y = x$ and $y = 3\sqrt{x}$. We will, again, create a 3-dimensional solid by revolving this region around the y-axis.

(a) Visualize the solid you'll create when you revolve this region around the y-axis.

(b) Draw a single rectangle in your region standing perpendicular to the y-axis.

(c) Let's use this rectangle to visualize the kth slice of this 3-dimensional solid. What does the "face" of it look like?

(d) Find the area of the face of the kth slice.

(e) Set up and evaluate the integral representing the volume of this solid.

6.3.2 Re-Orienting our Rectangles

Definition 6.3.2 Volume by Shells. If $f(x)$ and $g(x)$ are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$ (with $a \geq 0$), then the volume of the solid formed when the region bounded between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is revolved around the y-axis is

$$V = 2\pi \int_{x=a}^{x=b} x\,(f(x) - g(x)) \; dx.$$

◊

Activity 6.3.3 Volume by Shells. Let's consider the region bounded by the curves $y = x^3$ and $y - x + 6$ as well as the line $y = 0$. You might remember this region from Activity 6.2.3. This time, we'll create a 3-dimensional solid by revolving the region around the x-axis

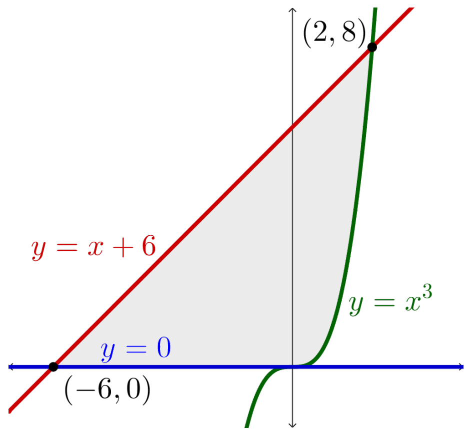


Figure 6.3.3

(a) Sketch one or two rectangles that are *perpendicular* to the x-axis. Then set up an integral expression to find the volume of the solid using them.

Hint. Note that in this context, we're actually using disks and washers. Also note that the bottom of the rectangles are bounded by $y = 0$ from $x = -6$ to $x = 0$ and then switches to being bounded by $y = x^3$ from $x = 0$ to $x = 2$.

(b) Now draw a single rectangle in the region that is *parallel* to the axis of revolution. Use this rectangle to visualize the kth slice of this 3-dimensional solid. What does that single rectangle create when it is revolved around the x-axis?

Hint. This won't create a disk or washer!

(c) Set up and evaluate the integral expression representing the volume of the solid.

6.3.3 Practice Problems

1. We say that the volume of a solid can be thought of as $\int_{x=a}^{x=b} A(x)\ dx$ where $A(x)$ is a function describing the cross-sectional area of our solid at an x-value between $x = a$ and $x = b$. Explain how this integral formula gets built, referencing the slice-and-sum (Riemann sum) method.

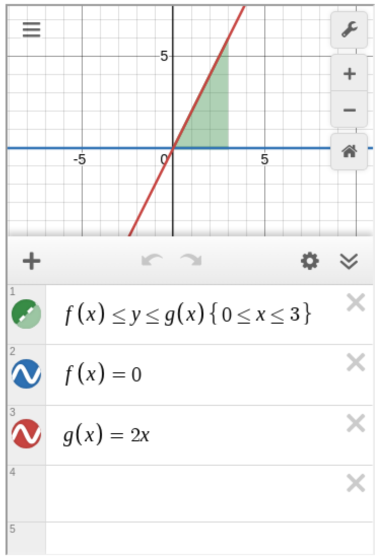
2. Explain the differences and similarities between the disk and washer methods for finding volumes of solids of revolution.

3. When do we integrate with regard to x (using a dx in our integral and writing our functions with x-value inputs) and when do we integrate with regard to y (using a dy in our integral and writing our functions with y-value inputs) when we're finding volumes using disks and washers? How do we know?

4. For each of the solids described below, set up an integral using the *disk/washer method* that describes the volume of the solid. It will be helpful to visualize the region, a rectangle on that region, as well as the rectangle revolved around the axis of revolution.

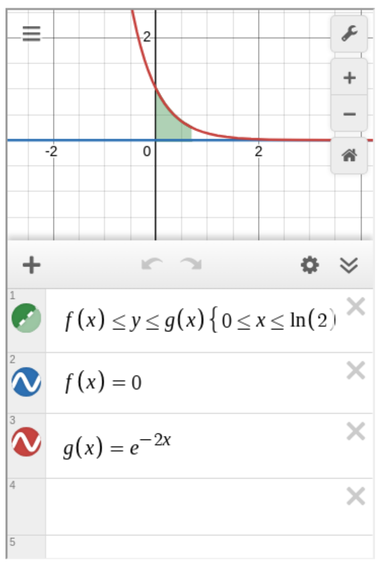
(a) The region bounded by the curve $y = 2x$ and the lines $y = 0$ and $x = 3$, revolved around the x-axis.

Hint.



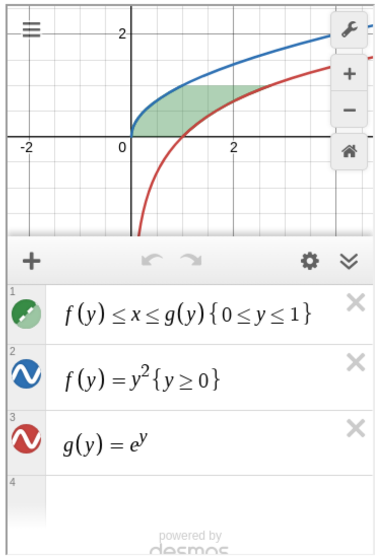
(b) The region bounded by the curve $y = e^{-2x}$ and the x-axis between $x = 0$ and $x = \ln(2)$, revolved around the x-axis.

Hint.



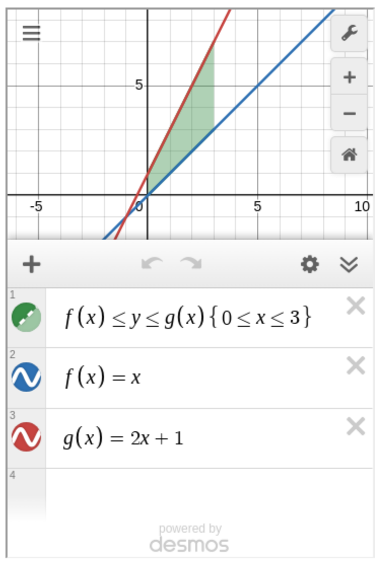
(c) The region bounded by the curves $y = \ln(x)$ and $y = \sqrt{x}$ between $y = 0$ and $y = 1$, revolved around the y-axis.

Hint.



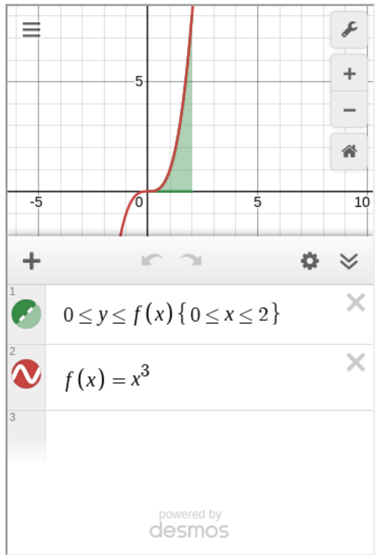
(d) The region bounded by the curves $y = 2x + 1$ and $y = x$ between $x = 0$ and $x = 3$, revolved around the x-axis.

Hint.



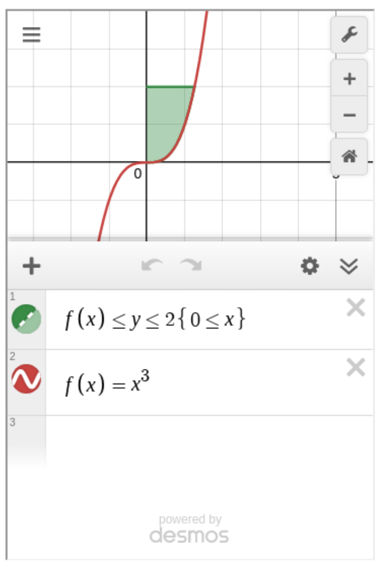
(e) The region bounded by the curve $y = x^3$, the x-axis, and the line $x = 2$, revolved around the y-axis.

Hint.



(f) The region bounded by the curve $y = x^3$ and the y-axis between $y = 0$ and $y = 2$, revolved around the y-axis.

Hint.



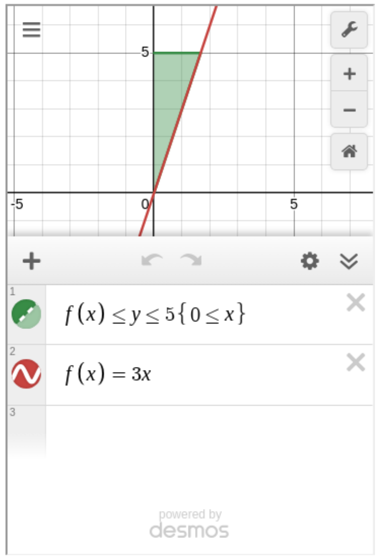
5. Explain where the pieces of the shell formula come from. How is this different than using disks/washers?

6. Say we're revolving a region around the x-axis to create a solid. Using the disk/washer method, we will integrate with respect to x. Using the shell method, we integrate with respect to y. Explain the difference, and why this difference occurs.

7. For each of the solids described below, set up an integral using the *shell method* that describes the volume of the solid. It will be helpful to visualize the region, a rectangle on that region, as well as the rectangle revolved around the axis of revolution.

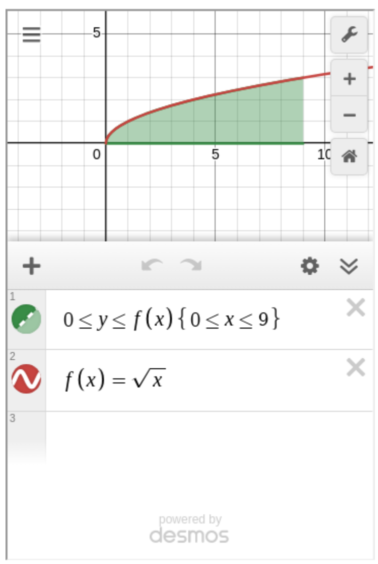
(a) The region bounded by the curve $y = 3x$ and the lines $x = 0$ and $y = 5$, revolved around the y-axis.

Hint.



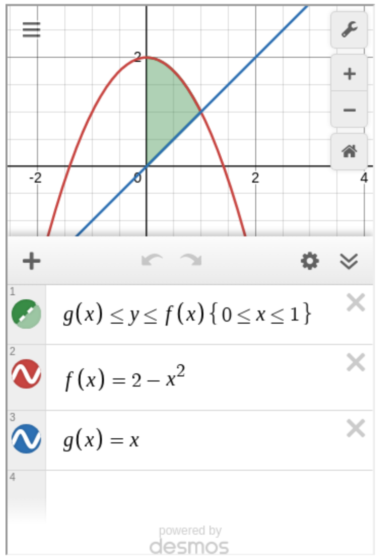
(b) The region bounded by the curve $y = \sqrt{x}$ and the x-axis between $x = 0$ and $x = 9$, revolved around the x-axis.

Hint.



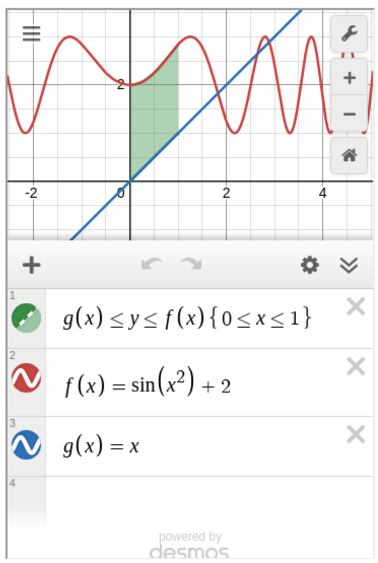
(c) The region bounded by the curves $y = 2 - x^2$ and $y = x$ and the line $x = 0$ revolved around the y-axis.

Hint.



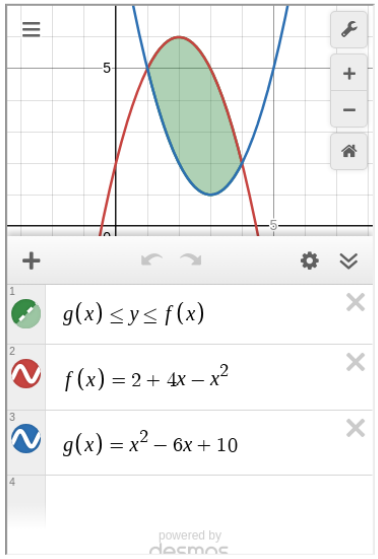
(d) The region bounded by the curves $y = sin(x^2) + 2$ and $y = x$ from $x = 0$ to $x = 1$, revolved around the y-axis.

Hint.



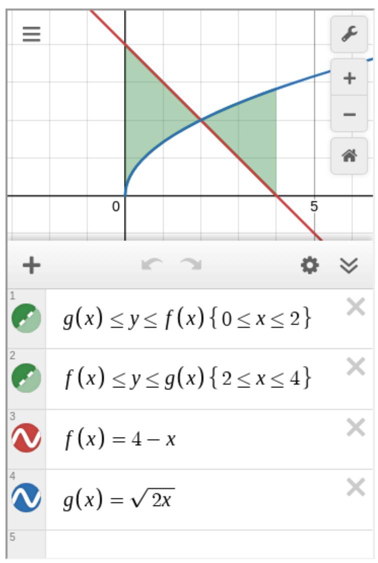
(e) The region bounded by the curves $y = x^2 - 6x + 10$ and $y = 2 + 4x - x^2$ revolved around the y-axis.

Hint.



(f) The region bounded by the curves $y = \sqrt{2x}$ and $y = 4 - x$ and the x-axis between $x = 0$ and $x = 4$, revolved around the x-axis.

Hint.



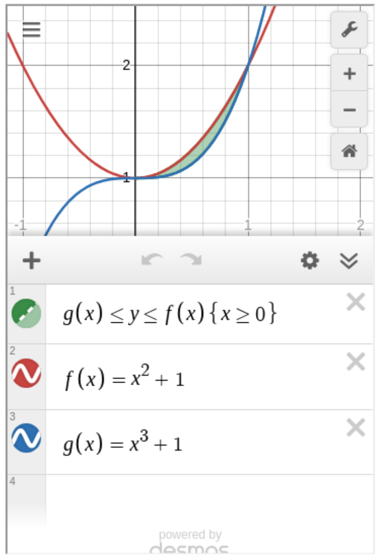
8. Pick at least 2 integrals from Exercise 4 to re-write using shells instead. What about those regions did you look for to choose which ones to re-write and which ones to not?

9. Pick at least 2 integrals from Exercise 7 to re-write using disks/washers instead. What about those regions did you look for to choose which ones to re-write and which ones to not?

10. For each of the following solids, set up an integral expression using either the disk/washer method or the shell method. You don't need to evaluate them, but you should do some careful thinking about how you set these up, especially as you choose between methods and what variable you are integrating with.

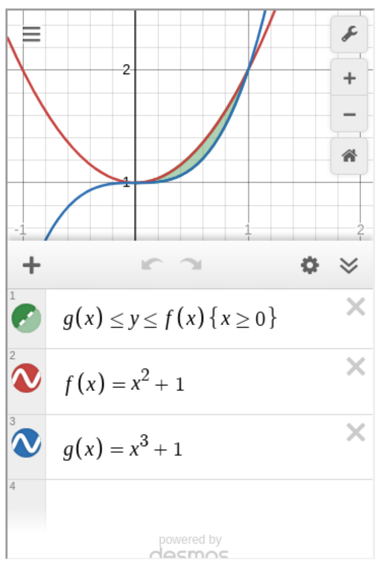
(a) The region bounded by the curves $y = x^2 + 1$ and $y = x^3 + 1$ in the first quadrant, revolved around the x-axis.

Hint.



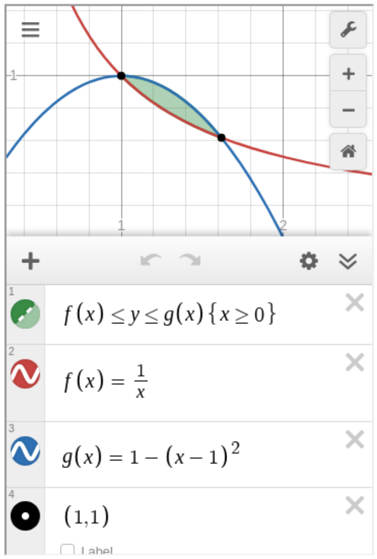
(b) The region bounded by the curves $y = x^2 + 1$ and $y = x^3 + 1$ in the first quadrant, revolved around the y-axis.

Hint.



(c) The region bounded by the curves $y = \frac{1}{x}$ and $y = 1 - (x - 1)^2$ in the first quadrant, revolved around the x-axis.

Hint.



6.4 More Volumes: Shifting the Axis of Revolution

We have introduced some methods for creating and calculating the volume of different 3 dimensional solids of revolution.

6.4.1 What Changes?

Let's first consider a volume created using disks or washers.

Activity 6.4.1 What Changes (in the Washer Method) with a New Axis? Let's revisit Activity 6.3.1 Volumes by Disks/Washers, and ask some more follow-up questions. First, we'll tinker with the solid we created: instead of revolving around the x-axis, let's revolve the same solid around the horizontal line $y = -3$.

(a) What changes, if anything, do you have to make to the rectangle you drew in Activity 6.3.1?

(b) What changes, if anything, do you have to make to the area of the "face" kth washer?

(c) What changes, if anything, do you have to make to the eventual volume integral for this solid?

Now let's consider a volume created using shells.

Activity 6.4.2 What Changes (in the Shell Method) with a New Axis? Let's revisit Activity 6.3.3 Volume by Shells, and ask some more follow-up questions about the shell method. Again, we'll tinker with the solid we created: instead of revolving around the x-axis, let's revolve the same solid around the horizontal line $y = 9$.

(a) What changes, if anything, do you have to make to the rectangle you drew in Activity 6.3.3?

(b) What changes, if anything, do you have to make to the area of the rectangle formed by "unrolling" up kth cylinder?

(c) What changes, if anything, do you have to make to the eventual volume integral for this solid?

In both of these cases, we can notice that the only changes we make are to the *radii*: we just need to re-measure the distance from axis of revolution to either the ends of the rectangle (in the washer method) or the side of the rectangle (in the shell method).

6.4.2 Formalizing These Changes in the Washers and Shells

Activity 6.4.3 More Shifted Axes. We're going to spend some time constructing *several* different volume integrals in this activity. We'll consider the same region each time, but make changes to the axis of revolution. For each, we'll want to think about what kind of method we're using (disks/washers or shells) and how the different axis of revolution gets implemented into our volume integral formulas.

Let's consider the region bounded by the curves $y = \cos(x) + 3$ and $y = \frac{x}{2}$ between $x = 0$ and $x = 2\pi$.

(a) Let's start with revolving this around the x-axis and thinking about the solid formed. While you set up your volume integral, think carefully about which method you'll be using (disks/washers or shells) as well as which variable you are integrating with regard to (x or y).

Hint. Note that in this region, we definitely want to use rectangles that stand up vertically. That means that they'll have a very small width, Δx, and sit perpendicular to the axis of revolution.

(b) Now let's create a different solid by revolving this region around the y-axis. Set up a volume integral, and continue to think carefully about which method you'll be using (disks/washers or shells) as well as which variable you are integrating with regard to (x or y).

Hint. We still will use the same tall rectangle with a small Δx side length, but this time it will be parallel to our axis of revolution.

(c) We'll start shifting our axis of revolution now. We'll revolve the same region around the horizontal line $y = -1$ to create a solid. Set up an integral expression to calculate the volume.

Hint 1. Note that we're still using the same rectangle (perpendicular to this horizontal axis), and so still integrating with regard to x, and using the washer method.

Hint 2. Since in the washer method our function outputs represent the radii, we need to re-measure the distance from our curves to the axis of revolution to find each circle's radius in the washer formula. For a y-value on each curve, how do we find the vertical distance down to the line $y = -1$?

(d) Now revolve the region around the line $y = 5$ to create a solid of revolution, and write down the integral representing the volume.

Hint. Note, now, that the y-value of the axis of revolution is larger than all of the y-values on the curves, meaning that to measure the distance from the axis of revolution to the curves, we might measure them in the opposite direction. Also, which curve is further away from the axis of revolution, representing the larger/outer radius?

(e) Let's change things up. Revolve the region around the vertical line $x = -1$ to create a new solid. Set up an integral representing the volume of that solid.

Hint 1. Note that the same rectangle that we used before is standing parallel to our axis of revolution. We're going to change methodology, and use the shell method!

Hint 2. Normally we use the input variable (x in this case) to measure the radius from the rectangles at different x-value to the axis of revolution, the y-axis. Now, though, we're not looking at the distance from x-values to $x = 0$. We're looking to find the radius, the distance from x-values in this region to $x = -1$.

(f) We'll do one more solid. Let's revolve this region around the line $x = 7$. Set up an integral representing the volume.

Hint. Note that this time, the axis of revolution's x-value is larger than all of the x-values in our region. So when we subtract to measure the radius, we need to subtract from $x = 7$ down to the varying x-values in the region.

6.4.3 Practice Problems

1. Consider the integral formula for computing volumes of a solid of revolution using the disk/washer method. What part of this integral formula represents the radius/radii of any circle(s)? Why is the radius represented using the function output from the curve(s) defining the region?

2. Consider the integral formula for computing volumes of a solid of revolution using the shell method. What part of this integral formula represents the radius/radii of any circle(s)? Why is the radius not represented using the function output from the curve(s) defining the region?

3. For each of the solids described below, set up an integral expression using disks/washers representing the volume of the solid.

(a) The region bounded by the curve $y = 1 - \sqrt{x}$ in the first quadrant, revolved around $x = 2$.

(b) The region bounded by the curve $y = 1 - \sqrt{x}$ in the first quadrant, revolved around $x = -1$.

(c) The region bounded by the curve $y = 1 - \sqrt{x}$ in the first quadrant, revolved around $y = -2$.

(d) The region bounded by the curve $y = 1 - \sqrt{x}$ in the first quadrant, revolved around $y = 3$.

4. For each of the solids described below, set up an integral expression using shells representing the volume of the solid.

(a) The region bounded by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant, revolved around the line $x = 2$.

(b) The region bounded by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant, revolved around the line $x = -1$.

(c) The region bounded by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant, revolved around the line $y = -2$.

(d) The region bounded by the curves $y = \sqrt{x}$ and $y = x$ in the first quadrant, revolved around the line $y = 3$.

6.5 Arc Length and Surface Area

Text of section.

6.6 Other Applications of Integrals

Text of section.

Chapter 7

Techniques for Antidifferentiation

7.1 Improper Integrals

Activity 7.1.1 Remembering a Theme so Far.

(a) Let's say that we want to find what the y-values of some function $f(x)$ are when the x-values are "infinitely close to" some value, $x = a$. Since there is no single x-value that is "infinitely close to" a that we can evaluate $f(x)$ at, we need to do something else. How do we do this?

(b) Let's say that we want to find the rate of change of some function instantaneously at a point with $x = a$. We can't find a rate of change unless we have two points, since we need to find some differences in the outputs and inputs. How do we do this?

(c) Suppose you want to find the total area, covered by an infinite number of infinitely thin rectangles. You have a formula for finding the dimensions and areas for some finite number of rectangles, but how do we get an infinite number of them?

(d) Can you find the common calculus theme in each of these scenarios?

Activity 7.1.2 Remembering the Fundamental Theorem of Calculus. We want to think about generalizing our notion of integrals a bit. So in this activity, section, we're going to think about some of the requirements for the Fundamental Theorem of Calculus and try to loosen them up a bit to see what happens. We'll try to construct meaningful approaches to these situations that fit our overall goals of calculating area under a curve.

This practice, in general, is a really good and common mathematical process: taking some result and playing with the requirements or assumptions to see what else can happen. So it might feel like we're just fiddling with the "What if?" questions, but what we're actually doing is good mathematics!

(a) What does the Fundamental Theorem of Calculus say about evaluating the definite integral $\displaystyle\int_{x=a}^{x=b} f(x)\,dx$?

(b) What do we need to be true about our setup, our function, etc. for us to be able to apply this technique to evaluate $\int_{x=a}^{x=b} f(x)\,dx$?

We are going to introduce the idea of "Improper Integrals" as kind-of-but-not-quite definite integrals that we can evaluate. They are going to violate the requirements for the Fundamental Theorem of Calculus, but we'll work to salvage them in meaningful ways.

Definition 7.1.1 Improper Integral. An integral is an **improper integral** if it is an extension of a definite integral whose integrand or limits of integration violate a requirement in one of two ways:

1. The interval that we integrate the function over is unbounded in width, or infinitely wide.
2. The integrand is unbounded in height, or infinitely tall, somewhere on the interval that we integrate over.

◊

Evaluating Improper Integrals (Infinite Width).

For a function $f(x)$ that is continuous on $[a, \infty)$, we can evaluate the improper integral $\int_{x=a}^{\infty} f(x)\ dx$:

$$\int_{x=a}^{\infty} f(x)\ dx = \lim_{t\to\infty} \int_{x=a}^{x=t} f(x)\ dx.$$

If $f(x)$ is continuous on $(-\infty, b]$, we can evaluate the improper integral $\int_{-\infty}^{x=b} f(x)\ dx$:

$$\int_{-\infty}^{x=b} f(x)\ dx = \lim_{t\to-\infty} \int_{x=t}^{x=b} f(x)\ dx.$$

Finally, if $f(x)$ is continuous on $(-\infty, \infty)$ and c is some real number, then we can evaluate the improper integral $\int_{-\infty}^{\infty} f(x)\ dx$:

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{t\to-\infty} \int_{x=t}^{x=c} f(x)\ dx + \lim_{t\to\infty} \int_{x=c}^{x=t} f(x)\ dx.$$

Evaluating Improper Integrals (Infinite Height).

For a function $f(x)$ that has an unbounded discontinuity (a vertical asymptote) at $x = c$

Definition 7.1.2 Convergence of an Improper Integral. We say that an improper integral **converges** if the limit of the appropriate definite integral exists. If the limit does not exist, then we say that the improper integral **diverges**. ◊

Practice Problems

1. Explain what it means for an integral to be improper. What kinds of issues are we looking at?

2. Give an example of an integral that is improper due to an unbounded or infinite interval of integration (infinite width).

3. Give an example of an integral that is improper due to an unbounded integrand (infinite height).

4. What does it mean for an improper integral to ``converge?'' How does this connect with limits?

5. What does it mean for an improper integral to ``diverge?'' How does this connect with limits?

6. Why do we need to use limits to evaluate improper integrals?

7. For each of the following improper integrals:

- Explain why the integral is improper. Be specific, and point out the issues in detail.
- Set up the integral using the correct limit notation.
- Antidifferentiate and evaluate the limit.
- Explain whether the integral converges or diverges.

(a) $\displaystyle\int_{x=0}^{\infty} \frac{1}{\sqrt{x+1}}\, dx$

(b) $\displaystyle\int_{x=0}^{\infty} e^{-2x}\, dx$

(c) $\displaystyle\int_{x=-1}^{x=3} \frac{1}{x+1}\, dx$

(d) $\displaystyle\int_{-\infty}^{x=0} \sqrt{e^x}\, dx$

(e) $\displaystyle\int_{x=2}^{x=8} \frac{5}{(x-2)^3}\, dx$

(f) $\displaystyle\int_{x=1}^{x=12} \frac{dx}{\sqrt[5]{12-x}}$

8. One of the big ideas in probability is that for a curve that defines a probability density function, the area under the curve needs to be 1. What value of k makes the function $\dfrac{kx}{(x^2+3)^{5/4}}$ a valid probability distribution on the interval $[0,\infty)$?

9. Let's consider the integral $\displaystyle\int_{x=1}^{\infty} \frac{\sqrt{x^2+1}}{x^2}\, dx$. This is a difficult integral to evaluate!

(a) First, compare $\sqrt{x^2+1}$ to $\sqrt{x^2}$ using an inequality: which one is bigger?

(b) Second, use this inequality to compare the function $\dfrac{\sqrt{x^2+1}}{x^2}$ to $\dfrac{1}{x}$ for $x > 0$: which one is bigger? Again, use your inequality from above to help!

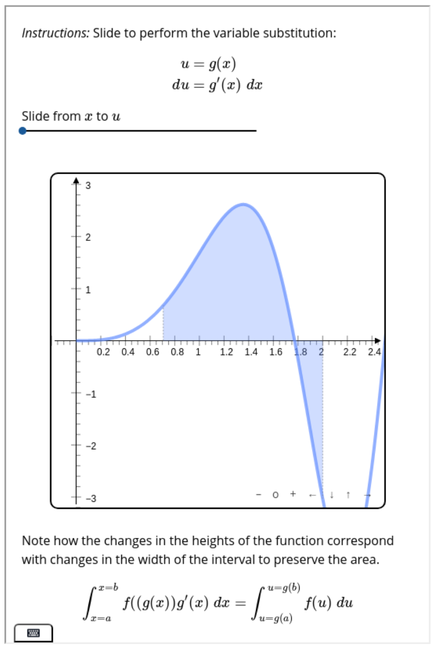
(c) Now compare $\int_{x=1}^{\infty} \frac{\sqrt{x^2+1}}{x^2}\, dx$ to $\int_{x=1}^{\infty} \frac{1}{x}\, dx$. Which one is bigger?

(d) Explain how we can use this result to make a conclusion about whether our integral, $\int_{x=1}^{\infty} \frac{\sqrt{x^2+1}}{x^2}\, dx$ converges or diverges.

7.2 u-Substitution

7.2.1 Undoing the Chain Rule

7.2.2 Substitution for Definite Integrals



7.2.3 More to Translate

Example 7.2.1 Integrate the following, making sure to translate the whole integrand function to be written in terms of u.

(a) $\int \left(\frac{x^3}{\sqrt{x^2+1}}\right)\, dx$

Hint. We can write x^3 as $x^2 \cdot x$, or if you *really* want to, we can write it as $\frac{1}{2}m^2 \cdot (2x)$

Solution.

$$\int \left(\frac{x^3}{\sqrt{x^2+1}}\right)\, dxu = x^2+1$$

$$du = 2x\ dx$$

$$\int \left(\frac{x^3}{\sqrt{x^2+1}}\right)\, dx = \frac{1}{2}\int \left(\frac{x^2 \cdot (2x)}{\sqrt{x^2+1}}\right)\, dx$$

$$= \frac{1}{2}\int \frac{(u-1)}{\sqrt{u}}\, du \qquad u = x^2+1 \leftrightarrow x^2 = u-1$$

$$
\begin{aligned}
&= \frac{1}{2}\int \frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\,du \\
&= \frac{1}{2}\int u^{1/2} - u^{-1/2}\,du \\
&= \frac{1}{2}\left(\frac{2u^{3/2}}{3} - 2u^{1/2}\right) + C \\
&= \frac{(x^2+1)^{3/2}}{3} - \sqrt{x^2+1} + C
\end{aligned}
$$

□

7.3 Manipulating Integrands to Reveal Substitution

We've looked at how to use a variable substitution to antidifferentiate composite functions. We've already seen, though, that sometimes identifying and actually using a helpful substitution can be difficult to do. In this section, we want to introduce some different strategies for noticing and setting up useful substitutions in some specific instances.

7.3.1 Rewriting the Integrand

Activity 7.3.1 A Negative Exponent. Let's think about this integral:

$$\int \frac{1}{1+e^{-x}}\,dx.$$

(a) Is there any composition in this integral? Pick it out, and either explain or show that using this to guide your substitution will not be helpful.

Hint. Notice that $-x$ is composed inside of the exponential function. Try a substitution with $u = -x$.

(b) What does e^{-x} mean? What does $\frac{1}{e^{-x}}$ mean?

(c) Re-write the integral, specifically focusing on the negative exponent. You should find that the function looks worse! How can you clean that up?

Hint 1. Re-write e^{-x} as $\frac{1}{e^x}$, giving you:

$$\int \frac{1}{1+\frac{1}{e^x}}\,dx.$$

Hint 2. Either add the fractions in the denominator or multiply the whole fraction by $\frac{e^x}{e^x}$.

Solution. You should have an integral that looks like:

$$\int \frac{e^x}{e^x+1}\,dx.$$

(d) Why is this new integral set up so much better for the purpose of u-substitution? How could we tell this just by looking at the initial integral?

Example 7.3.1

(a) $\int \frac{1}{x+x^{-1}}\, dx$

Hint. Try to re-write this integral as $\int \frac{x}{x^2+1}\, dx$.

Solution.

$$\int \frac{1}{x+x^{-1}}\, dx = \int \frac{x}{x^2+1}\, dx$$

$$u = x^2+1 \quad du = 2x\, dx$$

$$\begin{aligned}\int \frac{x}{x^2+1}\, dx &= \frac{1}{2}\int \frac{2x}{x^2+1}\, dx \\ &= \frac{1}{2}\int \frac{1}{u}\, du \\ &= \frac{1}{2}\ln|u| + C \\ &\frac{1}{2}\ln(x^2+1) + C\end{aligned}$$

□

7.3.2 Antidifferentiating Rational Functions

Strategies for antidifferentiating rational functions are just that: strategies. There aren't really easy ways to antidifferentiate these, but we'll find some common tactics to apply and try to build our intuition for noticing the different kinds of structure we can have in these rational functions. All of these strategies are based around cleverly re-writing our rational functions (using some algebraic manipulations) to reveal some structure. We'll try to notice the structure, so that we know what we're trying to reveal.

Integrating Rational Functions.

7.3.3 Practice Problems

1. Use polynomial division or some clever factoring to re-write and find the following indefinite integrals or evaluate the following definite integrals.

(a) $\int \left(\frac{x+4}{x-3}\right) dx$

(b) $\int \left(\frac{x^2+4}{x-4}\right) dx$

(c) $\int \left(\frac{t^2+t+6}{t^2+1}\right) dt$

(d) $\int_{x=2}^{x=4} \left(\frac{x^3+1}{x-1}\right) dx$

(e) $\int_{x=0}^{x=1} \left(\frac{x^4+1}{x^2+1}\right) dx$

2. Complete the square in order to find the following indefinite integrals.

(a) $\int \left(\frac{1}{x^2 - 2x + 10} \right) dx$

(b) $\int \left(\frac{x}{x^2 + 4x + 8} \right) dx$

(c) $\int \left(\frac{2x}{x^4 + 6x^2 + 10} \right) dx$

3. Find the following indefinite integrals.

(a) $\int \left(\frac{1}{x^{-1} + 1} \right) dx$

(b) $\int \left(\frac{\sin(\theta) + \tan(\theta)}{\cos^2(\theta)} \right) d\theta$

(c) $\int \left(\frac{1 - x}{1 - \sqrt{x}} \right) dx$

(d) $\int \left(\frac{1}{1 - \sin^2(\theta)} \right) d\theta$

(e) $\int \left(\frac{x^{2/3} - x^3}{x^{1/4}} \right) dx$

(f) $\int \left(\frac{4 + x}{\sqrt{1 - x^2}} \right) dx$

7.4 Integration By Parts

We've seen now that u-Substitution is a useful technique for undo-ing The Chain Rule. We set up the variable substitution with the specific goal of going backwards through the Chain Rule and antidifferentiating some composition of functions.

A reasonable next step is to ask: What other derivative rules can we "undo?" What other operations between functions should we think about? This brings us to Integration by Parts, the integration technique specifically for undo-ing Product Rule.

7.4.1 Discovering the Integration by Parts Formula

Activity 7.4.1 Discovering the Integration by Parts Formula. The product rule for derivatives says that:

$$\frac{d}{dx}\left(u(x) \cdot v(x)\right) = \underline{\hspace{3cm}} + \underline{\hspace{3cm}}.$$

We know that we intend to "undo" the product rule, so let's try to re-frame the product rule from a rule about derivatives to a rule about antiderivatives.

(a) Antidifferentiate the product rule by antidifferentiating each side of the equation.

$$\int \left(\frac{d}{dx}(u \cdot v)\right)\, dx = \int \underline{\hspace{3cm}} + \underline{\hspace{3cm}} dx$$

$$\underline{\hspace{3cm}} = \int \underline{\hspace{3cm}} dx + \int \underline{\hspace{3cm}} dx$$

Hint. Note that on the left side of this equation you're antidifferentiating a derivative. What will that give you? Then, on the right side, we're just splitting up the terms of the product rule into two different integrals.

(b) On the right side, we have two integrals. Since each of them has a product of functions (one function and a derivative of another), we can isolate one of them in this equation and create a formula for how to antidifferentiate a product of functions! Solve for $\int uv'\, dx$.

(c) Look back at this formula for $\int uv'\, dx$. Explain how this is really the product rule for derivatives (without just undo-ing all of the steps we have just done).

What made it so useful to pick $u = x$ instead of $dv = x\ dx$ in this case? Since we know that we are going to get another integral, one that specifically has the new derivative and new antiderivative that we find from the parts we picked, we noticed that differentiating the function x was much nicer than antidifferentiating it: we get a constant that we multiply by the trig function in this new integral, instead of a power function with an even bigger exponent. We can also notice that when it comes to the trig function, it doesn't really matter if we differentiate it or antidifferentiate it. In both cases, we get a $cos(x)$ in our new integral, with the only difference being whether it is positive or negative.

We typically use the substitutions $du = u'\ dx$ and $dv = v'\ dx$ to re-write the integrals.

Integration by Parts.

Suppose $u(x)$ and $v(x)$ are both differentiable functions. Then:

$$\int u\ dv = uv - \int v\ du.$$

When we select the parts for our integral, we are selecting a function to be labeled u and a function to be labeled as dv. We begin with one of the pieces of the product rule, a function multiplied by some other function's derivative. It is important to recognize that we do different things to these functions: for one of them, u, we need to find the derivative, du. For the other, dv, we need to find an antiderivative, b. Because of these differences, it is important to build some good intuition for how to select the parts.

7.4.2 Intuition for Selecting the Parts

Activity 7.4.2 Picking the Parts for Integration by Parts. Let's consider the integral:

$$\int x \sin(x)\ dx.$$

We'll investigate how to set up the integration by parts formula with the different choices for the parts.

(a) We'll start with selecting $u = x$ and $dv = \sin(x)\ dx$. Fill in the following with the rest of the pieces:

$$u = x \qquad v = \underline{\hspace{4cm}}$$
$$du = \underline{\hspace{4cm}} \qquad dv = \sin(x)\ dx$$

(b) Now set up the integration by parts formula using your labeled pieces. Notice that the integration by parts formula gives us another integral. Don't worry about antidifferentiating this yet, let's just set the pieces up.

Hint. $\int u\ dv = uv - \int v\ du$

(c) Let's swap the pieces and try the setup with $u = \sin(x)$ and $dv = x\ dx$. Fill in the following with the rest of the pieces:

$$u = \sin(x) \qquad v = \underline{\hspace{4cm}}$$
$$du = \underline{\hspace{4cm}} \qquad dv = x\ dx$$

(d) Now set up the integration by parts formula using this setup.

Hint. $\int u\ dv = uv - \int v\ du$

(e) Compare the two results we have. Which setup do you think will be easier to move forward with? Why?

Hint. When we say we need to keep moving forward with our setup, what we mean is that we have another integral to antidifferentiate. Which one will be easier to work with: $\int(-\cos(x))\ dx$ or $\int\left(\frac{x^2}{2}\cos(x)\right)$?

(f) Finalize your work with the setup you have chosen to find $\int x\sin(x)\ dx$.

What made things so much better when we chose $u = x$ compared to $dv = x\ dx$? We know that the new integral from our integration by parts formula will be built from the new pieces, the derivative we find from u and the antiderivative we pick from dv. So when we differentiate $u = x$, we get a constant, compared to antidifferentiating $dv = x\ dx$ and getting another power function, but with a larger exponent. We know this will be combined with a $cos(x)$ function no matter what (since the derivative and antiderivatives of $sin(x)$ will only differ in their sign). So picking the version that gets that second integral to be built from a trig function and a constant is going to be much nicer than a trig function and a power function. It was nice to pick x to be the piece that we found the derivative of!

Let's practice this comparison with another example in order to build our intuition for picking the parts in our integration by parts formula.

Activity 7.4.3 Picking the Parts for Integration by Parts. This time we'll look at a very similar integral:

$$\int x\ln(x)\ dx.$$

Again, we'll set this up two different ways and compsare them.

(a) We'll start with selecting $u = x$ and $dv = \ln(x)\ dx$. Fill in the following with the rest of the pieces:

$u = x$	$v =$ ________________
$du =$ ________________	$dv = \ln(x)\ dx$

Hint. You're not forgetting how to antidifferentiate $\ln(x)$. This is just something we don't know yet!

(b) Ok, so here we *have to* swap the pieces and try the setup with $u = \ln(x)$ and $dv = x\ dx$, since we only know how to differentiate $\ln(x)$. Fill in the following with the rest of the pieces:

$u = \ln(x)$	$v =$ ________________
$du =$ ________________	$dv = x\ dx$

(c) Now set up the integration by parts formula using this setup.

Hint. $\int u\ dv = uv - \int v\ du$

(d) Why was it fine for us to antidifferentiate x in this example, but not in Activity 7.4.2?

(e) Finish this work to find $\int x \ln(x)\ dx$.

Hint. Notice that $\left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right) = \frac{x}{2}$.

So here, we didn't actually get much choice. We couldn't pick $u = x$ in order to differentiate it (and get a constant to multiply into our second integral) since we don't know how to antidifferentiate $\ln(x)$ (yet: once we know how, it might be fun to come back to this problem and try it again with the parts flipped). But we can also notice that it ended up being fine to antidifferentiate x: the increased power from our power rule didn't really matter much when we combined it with the derivative of the logarithm, since the derivative of the log is *also a power function*! So we were able to combine those easily and actually integrate that second integral.

It is common for students to want to place functions into sort of hierarchy or classification guidelines for choosing the parts. Some students have found that the acronym LIPET (logs, inverse trig, power functions, exponentials, and trig functions) can be a useful tool for selecting the parts. When you have two different types of functions, it might help to select u to be whichever function shows up first in that list.

Example 7.4.1 Integrate the following:

(a) $\int x^2 e^x\ dx$

Hint. It doesn't matter whether we differentiate or antidifferentiate e^x, since we'll get the same thing. Let's pick $u = x^2$ so that we can differentiate it.

Solution.

$$\begin{array}{ll} u = x^2 & v = e^x \\ du = 2x\ dx & dv = e^x\ dx \end{array}$$

$$\int x^2 e^x\ dx = x^2 e^x - \int 2xe^x\ dx$$

We need to do more integration by parts!

$$\begin{aligned} u &= 2x & v &= e^x \\ du &= 2\ dx & dv &= e^x\ dx \end{aligned}$$

$$\begin{aligned} \int x^2 e^x\ dx &= x^2 e^x - \left(2xe^x - \int 2e^x\ dx \right) \\ &= x^2 e^x - 2xe^x + 2e^x + C \end{aligned}$$

(b) $\int 2x \tan^{-1}(x)\ dx$

Hint. We don't know how to antidifferentiate $\tan^{-1}(x)$, but we do know how to differentiate it!

Solution.

$$\begin{aligned} u &= \tan^{-1}(x) & v &= x^2 \\ du &= \tfrac{1}{x^2+1}\ dx & dv &= 2x\ dx \end{aligned}$$

$$\begin{aligned} \int 2x \tan^{-1}(x)\ dx &= x^2 \tan^{-1}(x) - \int \frac{x^2}{x^2+1}\ dx \\ &= x^2 \tan^{-1}(x) - \int \frac{(x^2+1)-1}{x^2+1}\ dx \quad \text{Alternatively, use long division.} \\ &= x^2 \tan^{-1}(x) - \int 1 - \frac{1}{x^2+1}\ dx \\ &= x^2 \tan^{-1}(x) - x + \tan^{-1}(x) + C \end{aligned}$$

□

7.4.3 Some Flexible Choices for Parts

We're going to look at a couple of examples where we can showcase some of the flexibility we have with our choices of parts. First, we'll revisit Task 7.4.1.b.In this example, when we got to that second integral, we noticed that for the fraction $\frac{x^2}{x^2+1}$, we could either do some long division (since the degrees in the numerator and denominator are the same) or do some clever re-writing of the numerator. Either way, we know that this fraction is *almost* 1...It's really $1\pm$ some bit (in this case, the extra bit was a fraction $\frac{1}{x^2+1}$).

What if we chose our parts differently? Not the u and dv parts, though, since we still haven't figured out how to antidifferentiate $\tan^{-1}(x)$. But we get one more choice!

Once we choose u, we don't really get a separate choice for du: it's simply the derivative of u with regard to x multiplied by the differential dx. But consider our choice of dv, and the subsequent process of finding v. Yes, there's only one possible answer, but in a much more real sense, there isn't just one possible answer. There are an infinite number of them! We know, due to the Mean Value Theorem and then later due to Theorem 4.1.2, that there are an infinite number of antiderivatives, all differing by at most a constant term. So let's pick a more appropriate antiderivative!

Example 7.4.2 Integrate $\int 2x \tan^{-1}(x)\ dx$, this time making a more intentional choice for v.

Hint. Note that if we pick $v = x^2 + 1$, then the second integral will be just delightful.

Solution.

$$\begin{array}{ll} u = \tan^{-1}(x) & v = x^2+1 \\ du = \frac{1}{x^2+1}\,dx & dv = 2x\,dx \end{array}$$

$$\begin{aligned}\int 2x\tan^{-1}(x)\,dx &= (x^2+1)\tan^{-1}(x) - \int \frac{x^2+1}{x^2+1}\,dx \\ &= x^2\tan^{-1}(x) + \tan^{-1}(x) - \int dx \\ &= x^2\tan^{-1}(x) + \tan^{-1}(x) - x + C\end{aligned}$$

□

So we get the same thing, but didn't have to think through the long division or the forced factoring. But the trade off here is that we almost *have to* see this coming to notice it. This flexibility doesn't always come into play for us. But we can look at a different kind of flexibility.

We've looked at integrals with both $\ln(x)$ and $\tan^{-1}(x)$. For these, and for other inverse functions specifically, we pick them to be the u part in our integration by parts problems because we don't know how do antidifferentiate them.

So let's look at $\int \ln(x)\,dx$, and we'll solve this integral by, specifically, differentiating $\ln(x)$ instead of antidifferentiating it.

Example 7.4.3 Antidifferentiating the Log Function. Integrate $\int \ln(x)\,dx$.

Hint. Pick $u = \ln(x)$, since we can differentiate it. What does that leave for dv?

Solution.

$$\begin{array}{ll} u = \ln(x) & v = x \\ du = \frac{1}{x}\,dx & dv = \,dx \end{array}$$

$$\begin{aligned}\int \ln(x)\,dx &= x\ln(x) - \int \frac{x}{x}\,dx \\ &= x\ln(x) - x + C\end{aligned}$$

□

We can use this same strategy to find antiderivatives of $\tan^{-1}(x)$, $\sin^{-1}(x)$, and eventually $\sec^{-1}(x)$.

For $\int \sec^{-1}(x)\,dx$, we'll need to use this same tactic of setting $u = \sec^{-1}(x)$ and $dv = \,dx$, but then later on we'll need to use a technique called Trigonometric Substitution to finish the problem.

Now that we know the antiderivative family for $\ln(x)$, we can revisit the problem in Activity 7.4.3, $\int x\ln(x)\,dx$, and try to work through the integration by parts when $u = x$ and $dv = \ln(x)\,dx$.

Example 7.4.4 Integrate $\int x\ln(x)\,dx$.

Solution.

$$\begin{array}{ll} u = x & v = x\ln(x) - x \\ du = \,dx & dv = \ln(x)\,dx \end{array}$$

$$\begin{aligned}\int x\ln(x)\,dx &= x(x\ln(x) - x) - \int x\ln(x) - x\,dx \\ & x^2\ln(x) - x^2 - \int x\ln(x)\,dx + \int x\,dx \\ & x^2\ln(x) - x^2 + \frac{x^2}{2} - \int x\ln(x)\,dx\end{aligned}$$

Note that this last integral is really recognizable: it's the one we started with! Let's "solve" this equation for that integral by adding it to both sides of our equation.

$$2\int x\ln(x)\,dx = x^2\ln(x) - \frac{x^2}{2}$$
$$\int x\ln(x)\,dx = \frac{x^2\ln(x)}{2} - \frac{x^2}{4} + C$$

□

7.4.4 Solving for the Integral

In this last example, we ended up seeing the original integral repeated when we did integration by parts. This is a useful technique, especially when we deal with functions that have a kind of "repeating" structure to their derivatives or antiderivatives. We'll look at a couple of classic integrals where we see this kind of technique employed.

Example 7.4.5 For each of the following integrals, use integration by parts to solve.

(a) $\int \sin(x)\cos(x)\,dx$

Hint. This one is pretty straight forward, since it doesn't really matter what we select as our parts. Notice, though, that this isn't the only way we can approach this! We can use u-substitution, or even re-write this using a trigonometric identity.

Solution.

$$\begin{aligned} u &= \sin(x) & v &= \sin(x) \\ du &= \cos(x)\,dx & dv &= \cos(x)\,dx \end{aligned}$$

$$\int \sin(x)\cos(x)\,dx = \sin^2(x) - \int \sin(x)\cos(x)\,dx$$
$$2\int \sin(x)\cos(x)\,dx = \sin^2(x)$$
$$\int \sin(x)\cos(x)\,dx = \frac{\sin^2(x)}{2} + C$$

(b) $\int e^x\cos(x)\,dx$

Solution.

$$\begin{aligned} u &= e^x & v &= \sin(x) \\ du &= e^x\,dx & dv &= \cos(x)\,dx \end{aligned}$$

$$\int e^x\cos(x)\,dx = e^x\sin(x) - \int e^x\sin(x)\,dx$$

$$\begin{aligned} u &= e^x & v &= -\cos(x) \\ du &= e^x\,dx & dv &= \sin(x)\,dx \end{aligned}$$

$$\int e^x\cos(x)\,dx = e^x\sin(x) - \int e^x\sin(x)\,dx$$
$$\int e^x\cos(x)\,dx = e^x\sin(x) + e^x\cos(x) - \int e^x\cos(x)\,dx$$
$$2\int e^x\cos(x)\,dx = e^x\sin(x) + e^x\cos(x)$$
$$\int e^x\cos(x)\,dx = \frac{e^x\sin(x) + e^x\cos(x)}{2} + C$$

□

Notice that we can come up with a bunch of different examples that are similar to Task 7.4.5.a. If we put trigonometric functions inside our integral, we'll have some options with how we approach them! We can use u-substitution, since the derivatives of trigonometric functions are other trigonometric functions. In Task 7.4.5.a, for instance, we could write $u = \sin(x)$ and $du = \cos(x)\,dx$, or even chose $u = \cos(x)$ and $du = -\sin(x)\,dx$.

The real issues will come when our integrand is not just a product of two trigonometric functions, but when they are products of trigonometric functions raised to exponents. We'll have some combinations of these products (which maybe makes us think about integration by parts) and composition (which points towards u-substitution). In the next section, we'll develop some strategies to deal with these kinds of integrals.

7.4.5 Practice Problems

1. Explain how we build the Integration by Parts formula, as well as what the purpose of this integration strategy is.

2. How do you choose options for u and dv? What are some good strategies to think about?

3. Let's say that you make a choice for u and dv and begin working through the Integration by Parts strategy. How can you tell if you've made a poor choice for your parts? Can you *always* tell?

4. Integrate the following.

(a) $\displaystyle\int 3x\sin(x)\,dx$

(b) $\displaystyle\int 5xe^x\,dx$

(c) $\displaystyle\int x^2e^{-x}\,dx$

(d) $\displaystyle\int x^2\ln(x)\,dx$

(e) $\displaystyle\int x^2\cos(x)\,dx$

(f) $\displaystyle\int x^3e^{-x}\,dx$

(g) $\displaystyle\int x\sin(x)\cos(x)\,dx$

(h) $\displaystyle\int e^x\sin(x)$

(i) $\displaystyle\int \sin^{-1}(x)\,dx$

(j) $\displaystyle\int \tan^{-1}(x)\,dx$

5. Evaluate the following definite integrals.

(a) $\displaystyle\int_{x=1}^{x=e} x\ln(x)\,dx$

(b) $\displaystyle\int_{x=0}^{x=\pi/4} x\cos(2x)\,dx$

(c) $\displaystyle\int_{x=0}^{x=\ln(5)} xe^x\,dx$

6. In this problem, we'll consider the integral $\int \sin^2(x)\,dx$. We'll integrate this in two different ways!

(a) We know that:

$$\int \sin^2(x)\,dx = \int \sin(x)\sin(x)\,dx.$$

Use the Integration by Parts strategy, and especially note that you can solve for the integral (Subsection 7.4.4 Solving for the Integral).

(b) We can use a trigonometric identity to re-write the integral:

$$\sin^2(x) = \frac{1-\cos(2x)}{2}.$$

So we have:

$$\int \sin^2(x)\,dx = \int \frac{1}{2} - \frac{1}{2}\cos(2x)\,dx.$$

Use u-substitution.

(c) Were your answers the same or different? Should they be the same? Why or why not? Are they connected somehow?

Hint. They might be different, but they should only be different by at most a constant.

7. For these next problems, we'll use $x = u^2$ and $dx = 2u\,du$ to substitute into the integral as written. Then use Integration by Parts.

(a) $\displaystyle\int \sin(\sqrt{x})\,dx$

(b) $\displaystyle\int e^{\sqrt{x}}\,dx$

7.5 Integrating Powers of Trigonometric Functions

Let's remind ourselves of two example problems that we've done in the past.

In Task 7.2.1.a, we performed a u-substitution, but needed to work to re-write our whole integrand in terms of u. Specifically, we found that in the numerator, there was an x^3, but $du = 2x\,dx$. We were substituting out a linear function of x in the numerator, but the actual function was cubic. This wasn't a problem: we re-wrote $x^3 = x^2 \cdot x$, and noticed that the extra x^2 was able to be substituted, since we could re-write out subsitution rule: we noted that $u = x^2 + 1$ is equivalent to $x^2 = u - 1$. This meant that even though we had an extra factor of x^2 "in" the part that we were using for substituting in the differential du, we were still able to translate the whole function to be written in terms of u.

Then, more recently, in Task 7.4.5.a, we noted that we could use a mix of methods to integrate this:

One on hand, we can look at the structure of the integrand and notice that we have a product of two functions! Integration by parts was a fine strategy to employ, and that's what we did in the example. On the other hand, we noticed that since we have this function-derivative pairing, a u-substitution was also appropriate.

In this section, we'll explore more combinations of trigonometric functions and build a strategy for antidifferentiating them that includes some ideas from both of these previous examples.

7.5.1 Building a Strategy for Powers of Sines and Cosines

Activity 7.5.1 Compare and Contrast. Let's do a quick comparison of two integrals, keeping the above examples in mind. Consider these two integrals:

$$\int \sin^4(x)\cos(x)\,dx \qquad \text{compared with} \qquad \int \sin^4(x)\cos^3(x)\,dx$$

(a) Consider the first integral, $\int \sin^4(x)\cos(x)\,dx$. Think about and set up a good technique for antidifferentiating. Without actually solving the integral, explain why this technique will work.

Hint. It might be helpful to notice that $\sin^4(x)$ can be re-written as $(\sin(x))^4$. Does this help reveal something important about the structure of this integrand?

(b) Now consider the second integral, $\int \sin^4(x)\cos^3(x)\,dx$. Does the same integration strategy work here? What happens when you apply the same thing?

Hint. Let $u = \sin(x)$ again, and $du = \cos(x)\,dx$. What happens with the cosine functions? How many are "left" after applying our substitution?

(c) We know that $\sin(x)$ and $\cos(x)$ are related to each other through derivatives (each is the derivative of the other, up to a negative). Is there some other connection that we have between these functions? We might especially notice that we have a $\cos^2(x)$ left over in our integral. Can we write this in terms of $\sin(x)$, so that we can write it in terms of u?

Hint. We have a trigonometric identity (the Pythagorean Identity):

$$\sin^2(x) + \cos^2(x) = 1.$$

(d) Why would this strategy not have worked if we were looking at the integrals $\int \sin^4(x)\cos^2(x)\,dx$ or $\int \sin^4(x)\cos^4(x)\,dx$? What, specifically, did we need in order to use this combination of substitution and trigonometric identity to solve the integral?

Integrating Powers of Sine and Cosine.

For integrals in the form $\int \sin^p(x)\cos^q(x)\,dx$ where p and q are real number exponents:

- If q, the exponent on $\cos(x)$ is odd, we should use $u = \sin(x)$ and $du = \cos(x)\ dx$. Then we can apply the Pythagorean Identity $\cos^2(x) = 1 - \sin^2(x)$.
- If p, the exponent on $\sin(x)$ is odd, we should use $u = \cos(x)$ and $du = -\sin(x)\ dx$. Then we can apply the Pythagorean Identity $\sin^2(x) = 1 - \cos^2(x)$.
- If both p and q are even, we can either use Integration by Parts or use the following power-reducing trigonometric identities:

$$\sin^2(x) = \frac{1-\cos(2x)}{2} = \frac{1}{2} - \frac{\cos(2x)}{2}$$
$$\cos^2(x) = \frac{1+\cos(2x)}{2} = \frac{1}{2} + \frac{\cos(2x)}{2}$$

A strange note, here, is that we typically pick our u-substitution based on looking to see a suitable candidate for u: we look for functions that are composed "inside" of other functions or we look for a function whose derivative is in the integral (the "function-derivative pair" that we talk about in Section 7.2). Here, though, we're selecting our substitution based on du: we're looking to see which function we can set aside one copy of for the differential, and then have an even power left over so that we can apply the Pythagorean Identity to translate the rest.

7.5.2 Building a Strategy for Powers of Secants and Tangents

Activity 7.5.2 Compare and Contrast (Again). We're going to do another Compare and Contrast, but this time we're only going to consider one integral:

$$\int \sec^4(x)\tan^3(x)\ dx.$$

We're going to employ another strategy, similar to the one for Integrating Powers of Sine and Cosine.

(a) Before you start thinking about this integral, let's build the relevant version of the Pythagorean Identity that we'll use. Our standard version of this is:

$$\sin^2(x) + \cos^2(x) = 1.$$

Since we want a version that connects $\tan(x)$, which is also written as $\frac{\sin(x)}{\cos(x)}$, with $\sec(x)$, or $\frac{1}{\cos(x)}$, let's divide everything in the Pythagorean Identity by $\cos^2(x)$:

$$\frac{\sin^2(x)}{\underline{\hspace{3cm}}} + \frac{\cos^2(x)}{\underline{\hspace{3cm}}} = \frac{1}{\underline{\hspace{3cm}}}$$

$$\underline{\hspace{4cm}} + \underline{\hspace{4cm}} = \underline{\hspace{4cm}}$$

Solution.

$$\tan^2(x) + 1 = \sec^2(x)$$

(b) Now start with the integral. We're going to use two different processes here, two different u-substitutions. First, set $u = \tan(x)$. Complete the substitution and solve the integral.

Hint. Here, $du = \sec^2(x)\,dx$. We'll also use $\sec^2(x) = \tan^2(x) + 1$.

Solution.

$$\begin{aligned}\int \sec^4(x)\tan^3(x)\,dx &= \int \underbrace{\sec^2(x)}_{\tan^2(x)+1}\tan^3(x)\sec^2(x)\,dx\\ &= \int (u^2+1)u^3\,du\\ &= \int u^5+u^3\,du\\ &= \frac{u^6}{6}+\frac{u^4}{4}+C\\ &= \frac{\tan^6(x)}{6}+\frac{\tan^4(x)}{4}+C\end{aligned}$$

(c) Now try the integral again, this time using $u = \sec(x)$ as your substitution.

Hint. Now $du = \sec(x)\tan(x)\,dx$, and we'll use the same Pythagorean identity, just re-written as $\tan^2(x) = \sec^2(x) - 1$.

Solution.

$$\begin{aligned}\int \sec^4(x)\tan^3(x)\,dx &= \int \sec^3(x)\underbrace{\tan^2(x)}_{\sec^2(x)-1}\sec(x)\tan(x)\,dx\\ &= \int u^3(u^2-1)\,du\\ &= \int u^5-u^3\,du\\ &= \frac{u^6}{6}-\frac{u^4}{4}+C\\ &= \frac{\sec^6(x)}{6}-\frac{\sec^4(x)}{4}+C\end{aligned}$$

(d) For each of these integrals, why were the exponents set up *just right* for u-substitution each time? How does the structure of the derivatives of each function play into this?

Hint. Notice we had an even exponent on the $\sec(x)$ function, but an odd exponent on the $\tan(x)$ function.

(e) Which substitution would be best for the integral $\int \sec^4(x)\tan^4(x)\,dx$. Why?

(f) Which substitution would be best for the integral $\int \sec^3(x)\tan^3(x)\,dx$. Why?

Integrating Powers of Secant and Tangent.

For integrals in the form $\int \sec^p(x)\tan^q(x)\,dx$ where p and q are real number exponents:

- If q, the exponent on $\tan(x)$, is odd, we can use $u = \sec(x)$ and $du = \sec(x)\tan(x)\,dx$. Then we can apply the Pythagorean Identity $\tan^2(x) = \sec^2(x) - 1$.

- If p, the exponent on $\sec(x)$, is even, we can use $u = \tan(x)$ and $du = \sec^2(x)\ dx$. Then we can apply the Pythagorean Identity $\sec^2(x) = \tan^2(x) + 1$.
- If p is odd and q is even, we can use Integration by Parts.

7.6 Trigonometric Substitution

Text of section.

7.7 Partial Fractions

Activity 7.7.1 Comparing Rational Integrands.

(a) Try to integrate the following:

$$\int \frac{9x+6}{x^2+3x-4}\ dx.$$

Explain what about this integral makes things very difficult.

Hint. Annoyingly, you can write this as:

$$\frac{9}{2}\int \frac{2x+3-\frac{5}{3}}{x^2+3x-4}\ dx = \frac{9}{2}\int \frac{2x+3}{x^2+3x-4}\ dx - \frac{15}{2}\int \frac{1}{x^2+3x-4}\ dx.$$

Now explain why the second integral is difficult.

(b) Confirm that

$$\frac{6}{x+4} + \frac{3}{x-1} = \frac{9x+6}{x^2+3x-4}.$$

(c) Try to integrate the following:

$$\int \frac{6}{x+4} + \frac{3}{x-1}\ dx.$$

(d) Which integral of the two would you rather integrate? Why?

Chapter 8

Infinite Series

8.1 Introduction to Infinite Sequences

8.1.1 Sequences as Functions

Before we move on to our actual goal of analyzing infinite series, we will construct infinite sequences. The big thing to remember here is that, when we build and analyze these sequences, we are are really building and analyzing functions. We want to keep this idea of sequences as functions in the forefront, since it will help us as we think about accumulating these function values into infinite series.

Activity 8.1.1 Building our First Sequences. We might already have some familiarity with sequences. Here, we'll focus less on some of the detailed mechanics and just think about these sequences as functions.

(a) Describe a sequence of numbers where you use a consistent rule/function to build each term (each number) based only on the *previous term* in the sequence. You will need to decide on some first term to start your sequence.

(b) Describe a different sequence of numbers using the same rule to generate new terms/numbers from the previous one. What do you need to do to make these two sequences different from each other?

(c) Describe a new sequence of numbers where you use a consistent rule/function to build each term based on its position in the sequence (i.e. the first term will be some rule/function based on the input 1, the second will be based on 2, you'll use 3 to get the third term, etc.). We will call the position of each term in the sequence the *index*.

(d) Describe another, new, sequence of numbers where you use a consistent rule/function to build each term based on its index. This time, make the terms get smaller in size as the index increases.

Definition 8.1.1 Explicit Formula. An infinite sequence defined using an **explicit formula** is one where the nth term of the sequence is defined as a function output of n, the term's index.

Using notation, we might say that $a_n = f(n)$ where:

- a is the ``name'' of the sequence (similar to how f and g are common names of functions).

- n is the index of the term, typically a non-negative integer.
- $f(n)$ is the function that we use to generate the terms.

◊

Definition 8.1.2 Recursion Relation. A sequence is defined using a **recursion relation** is one where the nth term of the sequence is defined as a function output of the previous term, the $(n-1)$st term. The sequence also needs some initial term to base the subsequent terms from.

Using notation, we might say that $a_n = f(a_{n-1})$. ◊

These definitions are relatively limited. You might, for instance, know of a *very* famous sequence that is defined recursively by having each term being the sum of the *two* previous terms. Our study of sequences will be brief and all pointing towards infinite series, so there are a lot of nuances about sequences that we will skip.

Activity 8.1.2 Returning to our First Sequences. Let's return back to the four sequences we created in Activity 8.1.1.

(a) For each of the sequences, how are we going to define them? Explicit formulas? Recursion relations? How do you know?

(b) Now, for each sequence, define the sequence formally using either an explicit formula or recursion relation, whichever matches with how you described the sequence in Activity 8.1.1.

Example 8.1.3 Practice Writing some Terms. For each of the following sequences, write out the first handful of terms. There isn't a set amount, but you should write out enough to get a feel for the sequence structure and how the different ways of defining the sequences work. In each, you can start the index n at 1 and count upwards $(n = 1, 2, 3, ...)$.

(a) $a_1 = \frac{1}{3}$ and $a_n = 2(a_{n-1})^2$

(b) $b_n = \frac{\sin(n)}{n^2}$

(c) $c_n = \sqrt[n]{n+1}$

(d) $d_n = \frac{n+e^n}{e^n}$

□

Activity 8.1.3 Describing These Sequences. Let's look at the sequences from Example 8.1.3. Go through the following tasks for each sequence.

(a) What do you think each sequence is "counting towards" (if anything)?

Hint. If you're not sure, maybe you need to write out a few more terms! You can also change how you write the numbers themselves: in some cases, fractions might be helpful, but in others it might be useful to write the numbers in decimal form. Maybe you'll approximate values of the sine or exponential functions, or maybe you'll leave them as $\sin(2)$ or e^3.

(b) Can you show that the sequence is counting towards what you think it is with a limit (or show that it's not counting towards anything)?

Hint. Some of these limits, as $n \to \infty$, will be tricky to work with! When might you want to use The Squeeze Theorem? When might you want to use L'Hopital's Rule?

Activity 8.1.4 Write the Sequence Rules. We'll look at some sequences by writing out the first handful of terms. From there, our goal is to write out more terms and eventually define each sequence fully.

For each sequence, write an explicit formula and a recursion relation to define the sequence. You can choose whether to start your index at $n = 0$ or $n = 1$.

(a) $\{a_n\} = \left\{4, \frac{2}{3}, \frac{1}{9}, \frac{1}{54}, ...\right\}$

Hint 1. It might be helpful to write these numbers using a common denominator! Or at least some of the numbers. Alternatively, you can try a common *numerator* (which is very fun to do, since we normally don't do that).

Hint 2. If you are recursively multiplying by a number each time, what will that look like in the explicit formula? How do we represent repeated multiplication?

(b) $\{b_n\} = \left\{\frac{3}{5}, \frac{2}{5}, \frac{5}{17}, \frac{3}{13}, \frac{7}{37}, ...,\right\}$

Hint 1. You can re-write these fractions! Have any of them been "reduced?"

Hint 2. Re-write $\frac{2}{5}$ and $\frac{3}{13}$ by scaling the numerator and denominator by 2. Can you find a formula for the numerator and denominator separately? This one is *very* difficult to find a recursion relation for, so feel free to only write it explicitly!

(c) $\{c_n\} = \left\{\frac{1}{5}, \frac{3}{5}, 1, \frac{7}{5}, ...\right\}$

Hint 1. This one will definitely be helpful to re-write so that all fractions have a common denominator.

Hint 2. If you are recursively adding something, how does that show up in the explicit formula? How do we repeatedly add?

(d) What kinds of connections do you notice between the explicit formulas and the recursion relations for these sequences?

Before moving on, we should think about a couple of notes:

- Notes about recursion and explicit formulas

8.1.2 Graphing Sequences

We have tried introducing and talking about sequences as special types of functions, mapping natural number inputs to real number outputs. If we are committed to thinking about sequences as functions, with maybe some special context, then we should really investigate how one of our primary representations of functions (graphs) manifests itself in this new context.

There really is not too much to think about here! We can focus on the domain of these functions. If we define a sequence $\{a_n\}$ explicitly, then we have some function $a_n = f(n)$, and we can plot this sequence function in the same way that we normally would any other function $g(x)$. We will use the horizontal axis for the inputs and the vertical axis to represent the outputs, and try to visualize the graph as the set of all of the pairs of inputs with their (single) corresponding output.

The only new feature, then, is that these functions have only non-negative integer inputs. So when we plot the points, we do not get some nice curve acting as a visual representation of the function: we get discrete points floating on the 2-dimensional plane, each with some consistent horizontal spacing between them.

Graph comparison.

Let's continue to think about these sequences as just functions in a special kind of context. How does this discrete context change how we talk about functions and what kinds of terminology we use?

8.1.3 Sequence Terminology

If a sequence is a function (and we're saying in this introductory section that it is), then we can think of all of the different terminology and adjectives that we use to describe functions. How many of them are relevant to sequences?

- Continuous?
- Differentiable?
- Integrable?
- Increasing?
- Decreasing?

For now, we'll talk about sequences in two ways: their direction and the size of their terms.

Definition 8.1.4 Direction of a Sequence. We say that a sequence $\{a_n\}_{n=1}^{\infty}$ is **increasing** if, for all $n = 1, 2, 3...$, $a_{n+1} > a_n$. If $a_{n+1} \geq a_n$ for all $n = 1, 2, 3, ...$ then we say that a_n is **non-decreasing**.

We say that a sequence $\{a_n\}_{n=1}^{\infty}$ is **decreasing** if, for all $n = 1, 2, 3...$, $a_{n+1} < a_n$. If $a_{n+1} \leq a_n$ for all $n = 1, 2, 3, ...$ then we say that a_n is **non-increasing**.

We say that a_n is constant if $a_{n+1} = a_n$, but this is a very boring sequence and we will likely not think terribly hard about these kinds of sequences.

Sometimes we might say that a sequence is **eventually non-increasing** if there is some $N > 1$, and the sequence is non-increasing for $n = N, N+1, N+2, ...$, and similarly for **eventually non-decreasing**. ♢

Definition 8.1.5 Monotonic Sequences. For the sequence $\{a_n\}_{n=1}^{\infty}$, we say that a_n is **monotonic** if a_n is either non-increasing or non-decreasing. ♢

Definition 8.1.6 Bounded Sequences. We say that a sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded below** if there is some real number M such that $a_n \geq M$ for all $n = 1, 2, 3, ...$.

Similarly we say that a sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded above** if there is some real number N such that $a_n \leq N$ for all $n = 1, 2, 3, ...$.

If a sequence has both an upper bound and a lower bound, then we often just say that the sequence is **bounded**. ♢

Lastly, we'll focus on the end-behavior of a sequence. We'll think about convergence of a sequence in the same way that we did for Improper Integrals: does the limit exist?

Definition 8.1.7 Sequence Convergence. For the sequence $\{a_n\}$, if L is some real number and $\lim_{n\to\infty} a_n = L$, the we say that the sequence $\{a_n\}$ **converges** to L. If this limit does not exist, we say that the sequence $\{a_n\}$ **diverges**. ♢

Theorem 8.1.8 Monotone Convergence Theorem. *If $\{a_n\}$ is a sequence that is both monotonic and bounded, then it must converge.*

This theorem seems to be a bit obvious to many students: why would we care about this, when we can just find a limit of the explicit formula for a sequence? We'll see throughout the rest of this chapter that this theorem is one of the most important and most useful results in our study of infinite sequences and infinite series. For now, though, let's use it to find the limits of some recursively defined sequences.

8.1.4 Some Cool Recursive Examples

Let's re-visit one of the recursively defined sequences that we've seen already and then think about a couple of other interesting ones. Before we do that, though, we should recognize why we need to treat recursively defined sequences a bit differently than ones defined explicitly.

In an explicit formula, the terms themselves are a function of n, the index. This means that we can simply apply a limit as $n \to \infty$ to understand whether or not the sequence converges and what it might converge to. These limits could be tricky, but we have the tools to evaluate them! In a recursion relation, though, each term is not a function of the index, which means we can't easily apply a limit as $n \to \infty$ to the term definition.

We'll be able to apply a limit, but it will feel a bit different: we're going to go into the limit work under the assumption that the limit exists. Let's see how it goes.

Example 8.1.9 Let's re-visit the first sequence from Example 8.1.3: $\{a_n\}_{n=1}^{\infty}$ where $a_1 = \frac{1}{3}$ and $a_n = 2(a_{n-1})^2$.

(a) Let's start by assuming that the sequence converges. That means that there exists some real number L such that

$$\lim_{n\to\infty} a_n = L.$$

What would this L be, if it exists? A key thing to note is that if $\lim_{n\to\infty} a_n$ exists (and we have a symbol, L, for it) then we can say that

$$\lim_{n\to\infty} a_{n-1} = L.$$

Whether or not this is obvious to you is not a mark of your understanding, but we need to make sure that this ends up being obvious to you. If it's not, that's ok! But it is an indicator that you should take a couple of minutes to think about this. Once you are convinced that these two limits are the same thing, move on to the next part.

(b) Let's now apply a limit to the sequence definition:

$$\begin{aligned}
a_n &= 2(a_{n-1})^2 \\
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} 2(a_{n-1})^2 \\
\underbrace{\lim_{n\to\infty} a_n}_{L} &= 2\left(\underbrace{\lim_{n\to\infty} a_{n-1}}_{L}\right)^2 \\
L &= 2L^2 \\
0 &= 2L^2 - L \\
0 &= L(2L-1)
\end{aligned}$$

And so we have two solutions to this equation: $L = 0$ and $L = \frac{1}{2}$. This is strange: how can a sequence have more than one value that it converges to?

It's because we have yet to take into account the initial term, a_1! Depending on this value, the sequence might converge or not, and if it does converge, then there are two options for what the sequence can converge to, based on the value of a_1.

(c) You can do the next part on your own, but I want you to pick different numbers for a_1 and write out some terms of the resulting sequence. You should find that some of them look like they're converging to 0, one of them will converge to $\frac{1}{2}$ (it's a fun hunt to find which one), and some will diverge.

Solution. You should find that if $a_1 = \frac{1}{2}$, then the sequence is constant and converges to $\frac{1}{2}$. If $0 \leq |a_1| < \frac{1}{2}$ then the sequence seems like it'll converge to 0. And if $|a_1| > \frac{1}{2}$, then it looks like the sequence diverges.

(d) Now it is up to us to show that this sequence, with $a_1 = \frac{1}{3}$, does converge. Sure, we have some evidence and a good conjecture that it converges to $\frac{1}{2}$, but that is just our good guess based on what we have seen in the first handful of numbers.

We will attempt to convince ourselves that this sequence is both monotonic and bounded. We'll begin with boundedness.

It should be clear that $a_n > 0$, since as long as $a_{n-1} \neq 0$, then $(a_{n-1})^2 > 0$. Since we start with $a_1 \neq 0$, we are guaranteed to get non-zero values from the formula for a new term! Great news, we have a lower bound.

Let's show that $\frac{1}{2}$ is an upper bound: $a_n < \frac{1}{2}$ when

$$\begin{aligned} 2(a_{n-1})^2 &< \frac{1}{2} \\ (a_{n-1})^2 &< \frac{1}{4} \\ a_{n-1} &< \frac{1}{2} \end{aligned}$$

Since $a_1 < \frac{1}{2}$, we know that each successive term will also be less than $\frac{1}{2}$. So we have an upper bound!

So the sequence $\{a_n\}$ is bounded. Now we just need to convince ourselves that this sequence is monotonic. We know that our terms are bounded above by $\frac{1}{2}$, and I hope that this means we can convince ourselves that since our terms are smaller than this, which would produce a constant sequence, then all of our terms are probably decreasing.

Let's show this by showing that $a_{n+1} - a_n < 0$:

$$\begin{aligned} a_{n+1} - a_n &= 2(a_n)^2 - a_n \\ &= a_n(2a_n - 1) \end{aligned}$$

We can solve for when this is negative! It shouldn't be hard to show that $a_n(2a_n - 1) < 0$ when $0 < a_n < \frac{1}{2}$. And we've already shown this is true in our case!

So $\{a_n\}$ is bounded and monotonic and must therefore converge because of the Monotone Convergence Theorem. Because $a_1 < \frac{1}{2}$, we know that this sequence doesn't converge to $\frac{1}{2}$, and so must converge to the only other option: 0.

□

There are some other fun ways of doing this same thing for other recursive examples. The argument above is relatively bulky to use, and so we understandably will not think about recursively defined sequences very much: we'll leave that topic for another course where we have more time to really explore them. If you are interested in trying this same argument with other sequences though, we'll end this section with two more fun examples.

Example 8.1.10

(a) Consider the sequence defined by $b_n = \sqrt{2 + b_{n-1}}$ with $b_1 = \sqrt{2}$. Does this sequence converge? To what?

Hint. Write out some terms to get a feel for things! Then, assuming that the sequence converges to some real number, L, think about what happens when you apply a limit as $n \to \infty$: we should get the equation $L = \sqrt{2 + L}$.

(b) Consider the sequence defined by $c_n = \frac{1}{2(c_{n-1})+1}$ with $c_1 = 1$. Does this sequence converge? To what?

Hint. Write out some terms to get a feel for things! Then, assuming that the sequence converges to some real number, L, think about what happens when you apply a limit as $n \to \infty$: we should get the equation $L = \frac{1}{2L+1}$.

□

8.2 Introduction to Infinite Series

Let's try to introduce the idea of an infinite series using a framework that we know and are (maybe) comfortable with: integrals!

With an integral, we have a nice way of evaluating integrals of nicely behaved functions with finite limits of integration (Fundamental Theorem of Calculus (Part 2)).

Then, when we talked about improper integrals, we built a nice way to think about evaluating integrals with unbounded limits of integration (Evaluating Improper Integrals (Infinite Width)). How will we use this to think about infinite series, a sum of the infinitely many terms from an infinite sequence?

8.2.1 Partial Sums

If we approach infinite series in a manner similar to improper integrals, then we will need to do a couple of things.

1. Truncate the infinite series at some finite ending point. This is what we did with the integral, when we replaced the infinity with some real number variable t. We might use n for the series ”ending index.”

2. Find a formula for this truncated/finite version. For the integrals, we could use the Fundamental Theorem of Calculus (Part 2) for this! For series, we'll need to do something else.

3. Apply a limit as t (or n in the case of infinite series) goes off to infinity!

Activity 8.2.1 How Do We Think About Infinite Series? Let's consider the following sequence:

$$\left\{\frac{9}{10^n}\right\}_{n=1}^{\infty}$$

(a) White out the first 5 terms of the sequence.

(b) What does this sequence converge to? Show this with a limit!

(c) Now we'll construct a new sequence, this time by adding things up. We're going to be working with the sequence $\{S_n\}_{n=1}^{\infty}$ where

$$S_n = \sum_{k=1}^{k=n} \left(\frac{9}{10^k}\right).$$

Write out the first five terms of this sequence: S_1, S_2, S_3, S_4, S_5.

(d) Can you come up with an explicit formula for S_n?

(e) Does $\{S_n\}$ converge or diverge? Use a limit to find what it converges to!

(f) What do you think this means for the infinite series $\sum_{k=1}^{\infty}\left(\frac{9}{10^k}\right)$? Does the infinite series converge or diverge?

This is hopefully a nice little introduction to how we'll think about infinite series: we'll consider, instead, the sequence of sums where we add up more and more terms. This is also a nice first example, because we really just showed that

$$0.999... = 1$$

since

$$\sum_{k=1}^{\infty}\left(\frac{9}{10^k}\right) = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + ...$$
$$= 0.9 + 0.09 + 0.009 + ...$$
$$1.$$

But more importantly, we now have a good strategy for thinking about infinite series as sequences of *partial sums*.

Definition 8.2.1 Partial Sum. For an infinite series $\sum_{k=1}^{\infty} a_k$, we call $S_n = \sum_{k=1}^{n} a_k$ the nth **Partial Sum** of the infinite series. ◊

Definition 8.2.2 Series Convergence. We say that the infinite series $\sum_{k=1}^{\infty} a_k$ **converges** to the real number L if the sequence $\{S_n\}_{n=1}^{\infty}$ converges to L ($\lim_{n\to\infty} S_n = L$), where $S_n = \sum_{k=1}^{n} a_k$ is the nth partial sum of the infinite series.

If the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ diverges (the limit $\lim_{n\to\infty} S_n$ does not exist), then we say that the infinite series $\sum_{k=1}^{\infty} a_k$ **diverges**. ◊

8.2.2 Visualizing the Sequence of Partial Sums

Since we'll think about an infinite series $\sum_{k=0}^{\infty} a_k$ as the sequence of its partial sums, $\{\sum_{k=0}^{n} a_k\}$, then we can think about visualizing an infinite series as really the same thing as visualizing a sequence in general (Subsection 8.1.2).

Graphing examples.

8.2.3 Special Series

Let's look at three examples where we can think about partial sums and play with our new idea of series convergence.

Example 8.2.3 For each of the following series, write out a few of the terms of the series. Then write out the corresponding partial sums. Use these to find a formula for S_n, the nth partial sum. Then make a claim about whether or not the series converges and what it converges to.

(a) $\sum_{k=0}^{\infty}\left(\frac{1}{2^k}\right)$

Solution.

$$\begin{aligned}
S_0 &= 1\\
S_1 &= 1+\frac{1}{2}=\frac{3}{2}\\
S_2 &= 1+\frac{1}{2}+\frac{1}{4}=\frac{7}{4}\\
S_3 &= 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{15}{8}\\
S_n &= 1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^n}=\frac{2^{n+1}-1}{2^n}\\
&= 2-\frac{1}{2^n}\\
\lim_{n\to\infty} S_n &= \lim_{n\to\infty}\left(2-\frac{1}{2^n}\right)\\
&= 2
\end{aligned}$$

The series converges to 2.

(b) $\sum_{k=2}^{\infty}\left(\frac{k}{\ln(k)}-\frac{k+1}{\ln(k+1)}\right)$

Solution.

$$\begin{aligned}
S_2 &= \frac{2}{\ln(2)}-\frac{3}{\ln(3)}\\
S_3 &= \frac{2}{\ln(2)}\underbrace{-\frac{3}{\ln(3)}+\frac{3}{\ln(3)}}_{0}-\frac{4}{\ln(4)}\\
&= \frac{2}{\ln(2)}-\frac{4}{\ln(4)}\\
S_4 &= \frac{2}{\ln(2)}\underbrace{-\frac{3}{\ln(3)}+\frac{3}{\ln(3)}}_{0}\underbrace{-\frac{4}{\ln(4)}+\frac{4}{\ln(4)}}_{0}-\frac{5}{\ln(5)}\\
&= \frac{2}{\ln(2)}-\frac{5}{\ln(5)}\\
S_5 &= \frac{2}{\ln(2)}\underbrace{-\frac{3}{\ln(3)}+\frac{3}{\ln(3)}}_{0}\underbrace{-\frac{4}{\ln(4)}+\frac{4}{\ln(4)}}_{0}\underbrace{-\frac{5}{\ln(5)}+\frac{6}{\ln(6)}}_{0}-\frac{6}{\ln(6)}\\
&= \frac{2}{\ln(2)}-\frac{6}{\ln(6)}\\
S_n &= \frac{2}{\ln(2)}\underbrace{-\frac{3}{\ln(3)}+\frac{3}{\ln(3)}}_{0}\underbrace{-\frac{4}{\ln(4)}+\frac{4}{\ln(4)}}_{0}\underbrace{-\frac{5}{\ln(5)}+...+\frac{n}{\ln(n)}}_{0}-\frac{n+1}{\ln(n+1)}
\end{aligned}$$

$$= \frac{2}{\ln(2)} - \frac{n+1}{\ln(n+1)}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2}{\ln(2)} - \frac{n+1}{\ln(n+1)}$$

$$= -\infty$$

So the series $\sum_{k=2}^{\infty} \left(\frac{k}{\ln(k)} - \frac{k+1}{\ln(k+1)}\right)$ diverges.

(c) $\sum_{k=2}^{\infty} \left(\frac{2}{k^2-1}\right)$

Hint. This one is tricky! It's hard to notice anything unless we write out the series term formula a bit differently. Use Partial Fractions to re-write $\frac{2}{k^2-1}$ as $\frac{1}{k-1} - \frac{1}{k+1}$.

Solution.

$$\sum_{k=2}^{\infty} \left(\frac{2}{k^2-1}\right) = \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k+1}\right)$$

$$S_2 = 1 - \frac{1}{3}$$

$$S_3 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4}$$

$$S_4 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5}$$

$$= 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}$$

$$S_5 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6}$$

$$= 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}$$

$$S_n = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \ldots + \underbrace{\frac{1}{n-2} - \frac{1}{n}}_{a_{n-1}} + \underbrace{\frac{1}{n-1} - \frac{1}{n+1}}_{a_n}$$

$$1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+2}\right)$$

$$= \frac{3}{2}$$

□

These examples are a bit misleading: we often won't be able to do this kind of thing! For most infinite series, we will struggle to find an explicit formula for the nth partial sum. In these examples, though, we took advantage of some specific structure.

In this first example (as well as the example in Activity 8.2.1), we noticed that because of the exponential function defining the terms, we were able to find some nice patterns in the partial sums. We'll explore this a bit more later in Section 8.6.

Then in these other two examples, we noticed that once we could write each term as really a difference of two fractions that have a really similar structure, we got these "repeat" values from term to term where the opposite signs made things add up to 0. These are called "telescoping series," and they're mostly fun examples to think about partial sum formulas. We'll see some pop up later though, and Partial Fraction Decomposition is a nice trick to keep in mind for these kinds of things.

8.3 The Divergence Test and the Harmonic Series

8.3.1 The Relationship Between a Sequence and Series

We have looked at both infinite sequences and infinite series so far, and, to make things complicated, we're really thinking about an infinite series (of terms from an infinite sequence) as an infinite sequence (of partial sums of the series). We've looked at how to visualize these (in both Subsection 8.1.2 and Subsection 8.2.2).

Let's first start with defining a new series. This is a relatively important one by itself (it *does* have its own name), but it's mostly an important series because it leads us into some new and interesting ways of thinking about series in general.

Definition 8.3.1 Harmonic Series. We call the series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

the **Harmonic Series**. ◊

Activity 8.3.1 Investigating the Harmonic Series.

(a) Write out the first several terms of the harmonic series, terms from $\left\{\frac{1}{k}\right\}_{k=1}^{\infty}$. Write however many you need to get a feel for how the terms work.

(b) Can you find out how many terms you would have to go "into" the series before the term was less than 0.00000001?

Hint. When is $\frac{1}{k} < \frac{1}{10^8}$?

(c) Can you do this same kind of thing, no matter how small? For instance, how many terms would you have to go into the series before the term was less than some real number ε where $\varepsilon > 0$?

Hint. When is $\frac{1}{k} < \varepsilon$?

(d) Remind/explain/convince yourself that what we've really done is show that $\lim_{k\to\infty} \frac{1}{k} = 0$. This isn't a new or terribly interesting fact, but make sure that you understand why the argument above shows this.

(e) Let's do something very similar, but with $\left\{\sum_{k=1}^{n} \frac{1}{k}\right\}_{n=1}^{\infty}$, the sequence of partial sums, instead. Write out the first few partial sums. There's no specific number that you *need* to write, but make sure to write enough partial sums to get a feel for how the partial sums work.

(f) Can you find out how many terms you need to add up until the partial sum is larger than 1?

Hint. Find a value for n to give

$$\sum_{k=1}^{n} \frac{1}{k} > 1.$$

(g) Can you find out how many terms you need to add up until the partial sum is larger than 5?

Hint. Find a value for n to give

$$\sum_{k=1}^{n} \frac{1}{k} > 5.$$

Solution.

$$\sum_{k=1}^{83} \frac{1}{k} \approx 5.00207...$$

This is the first partial sum greater than 5.

(h) Can you find out how many terms you need to add up until the partial sum is larger than 10?

Hint. Find a value for n to give

$$\sum_{k=1}^{n} \frac{1}{k} > 10.$$

This will be absolutely awful to try calculating by hand! Use some piece of technology!

Solution.

$$\sum_{k=1}^{12367} \frac{1}{k} \approx 10.000043...$$

This is the first partial sum greater than 10.

(i) Do you think that for any positive number S, we can always find some partial sum $\sum_{k=1}^{n} \frac{1}{k} > S$? What do you think this would mean about

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k}?$$

To actually show that for any $S > 0$ we could always find an $n > 1$ where

$$\sum_{k=1}^{n} \frac{1}{k} > S$$

is an extremely difficult task! We will show that the Harmonic Series diverges in a different way, but for now I want us to notice these contradictory results: we have a series whose terms get small, but whose partial sums do not seem to converge.

We have $\frac{1}{k} \to 0$ but it seems like $\lim_{n\to\infty} S_n$ does not exist. Is this behavior special to the Harmonic Series? Is this something we should make note of? Is there some other connection between the terms of a series and the behavior of the partial sums of the series that we need to note?

Let's continue to think about this strange series, but actually prove that the series itself diverges.

Theorem 8.3.2 The Harmonic Series Diverges. *The* ***Harmonic Series****,* $\sum_{k=1}^{\infty} \frac{1}{k}$*, diverges.*

Proof. Let's assume, for the sake of eventual contradiction, that the harmonic series converges. ■

8.3.2 The Divergence Test

Theorem 8.3.3 Divergence Test. *For an infinite series $\sum_{k=0}^{\infty} a_k$, if the infinite series converges then*

$$\lim_{k\to\infty} a_k = 0.$$

This is equivalent to saying that if

$$\lim_{k\to\infty} a_k \neq 0$$

then the infinite series $\sum_{k=0}^{\infty} a_k$ diverges.
Proof. ■

Example 8.3.4 Apply the Divergence Test to the following series and interpret the results.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{k^15 - 4k^10 + 10k^4}{e^{2k}}$

Hint. We can do a couple of things here! There is a nice result about limits of polynomials that we can use in the numerator (Polynomial End Behavior Limits). We could also get this same result using some other techniques, like what we use to prove that theorem. Then we can use L'Hopital's Rule to evaluate the limit, since we have a $\frac{\infty}{\infty}$ indeterminate form.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2+1}$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt[k]{k}}$

□

8.4 The Integral Test

8.4.1 Infinite Series As a Kind of Integral

We start here with a connection between objects. Earlier (in Subsection 8.1.1) we tried to describe sequences as just a special kind of function: the domain is the set of non-negative integer (or positive integers, depending on whether we start our index at $n = 0$ or $n = 1$) and we map these inputs to real number outputs. And now we want to think about what it might mean to accumulate the values of these kinds of functions.

Function value accumulation is what we've been looking at lately! That's what integration is! We are trying to accumulate all of the function values and weigh them based on their "width." In the context of continuous functions, that means we start approximating this accumulation by looking at some finite number of function values that we pick, and we give them some Δx width between them. That's our Riemann sum:

$$\sum_{k=1}^{n} f(x_k^*)\ \Delta x$$

And from there, we work on making that space between function values get smaller (as the number of function values we use gets higher). So when n is the number of function values, we can let $n \to \infty$ and correspondingly we get $\Delta x \to dx$, the differential in our integral:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\ \Delta x = \int_{x=a}^{x=b} f(x)\ dx.$$

And this is how we've talked about infinite series so far, even adopting the same notions of convergence and thinking about how we extend a familiar idea (in this case adding numbers, compared to integrating a function) out to infinity: we just keep walking our (finite) ending point out to infinity using a limit!

So this brings us to this comparison of the same types of objects across these two different contexts.

Table 8.4.1 Comparisons of Calculus Objects in Continuous and Discrete Contexts

Object	Continuous Context	Discrete Context
Function	$f(x)$	a_k
Graph		
Finite Accumulation	Definite Integral $A(t) = \int_{x=0}^{x=t} f(x)\ dx$	Partial Sum $S_n = \sum_{k=0}^{n} a_k$
Infinite Accumulation	Improper Integral $\int_{x=0}^{\infty} f(x)\ dx$	Infinite Series $\sum_{k=0}^{\infty} a_k$

So in this section, we'll investigate this link between infinite series and improper integrals as the same kind of object occurring in different contexts. Intuitively, then, they'll be related to each other, under the right conditions.

8.4.2 The Integral Test

We'll build the integral test.

Theorem 8.4.2 Integral Test. *If $\sum_{k=0}^{\infty} a_k$ is an infinite series with $a_k > 0$ for all $k \geq 0$ and $f(x)$ is a continuous and decreasing function with $f(k) = a_k$ for all $k \geq 0$, then we can compare the behaviors of $\sum_{k=0}^{\infty} a_k$ and $\int_{x=0}^{\infty} f(x)\ dx$: the integral and the series are guaranteed to either both diverge or both converge.*

8.4.3 Why Do We Need These Conditions?

Riemann sum approximation Oscillating function can make an integral converge but series diverge $\sin^2(\pi(x))$ opposite.

8.5 Alternating Series and Conditional Convergence

Before we move too far forward, let's circle back to a point made in Subsection 8.4.3. In the Integral Test, we required the terms of our series (and the continuous function we connected it with) to be positive. This was really just a mechanism that allowed us to say, in our proof, that the sequence of partial sums was monotonic. When we accumulate more of a positive thing, the total gets bigger. This is half of what we needed for us to employ the Monotone Convergence Theorem. Because this is such a useful tool, we'll see more of this "positive term series" condition showing up in the tools we use to see if a series converges.

But that makes this a perfect time to stop and ask a hallowed mathematical question: *What happens if that property isn't there?* What happens when our series does not only have positive terms?

We definitely have fewer tools to use, since we don't get anything that relies on applying the Monotone Convergence Theorem to partial sums. So instead, we'll take a brief detour into something we call **Alternating Series** (a series whose terms alternate in sign).

Activity 8.5.1 Which is More Likely to Converge? We're going to try to think about what might be different when we analyze an alternating series compared to a series with only positive (or non-negative) terms.

Let's say that $\{a_k\}_{k=0}^{\infty}$ is some sequence of positive real numbers. Now let's consider the two series:

$$\sum_{k=0}^{\infty} a_k \qquad \text{vs} \qquad \sum_{k=0}^{\infty} (-1)^k a_k$$

(a) Let's first consider the sequences of terms: $\{a_k\}$ compared with $\{(-1)^k a_k\}$. Is either of these more or less likely to converge? Does this tell us anything about whether or not the corresponding series converges?

Hint 1. Try thinking about how we might find $\lim_{k\to\infty}(-1)^k a_k$, especially using the The Squeeze Theorem.

Hint 2. What does the Divergence Test say? Is either of these sequences more or less likely to converge to 0 (or not)?

(b) Now let's think of the partial sums:

$$\left\{\sum_{k=0}^{n} a_k\right\}_{n=0}^{\infty} \qquad \text{vs} \qquad \left\{\sum_{k=0}^{n} (-1)^k a_k\right\}_{n=0}^{\infty}$$

Is either of these sequences more or less likely to converge? Does this tell us anything about whether or not the corresponding series converges?

Definition 8.5.1 Alternating Series. An infinite series $\sum_{k=0}^{\infty} a_k$ is called an **Alternating Series** when $a_k = (-1)^k |a_k|$ or $a_k = (-1)^{k+1}|a_k|$ for all $k = 0, 1, 2, \ldots$. That is, the sign of the terms alternates:

$$\sum_{k=0}^{\infty} a_k = |a_0| - |a_1| + |a_2| - + \ldots$$

◊

Theorem 8.5.2 Alternating Series Test. *If $\sum_{k=0}^{\infty} a_k$ is an alternating series and the size of the terms $|a_k|$ is decreasing, then if $\lim_{k\to\infty} a_k = 0$ then $\sum_{k=0}^{\infty} a_k$ converges.*

8.6 Common Series Types

In this section, we'll stop and recap some of the common series types that we should recognize moving forward. We'll look at the structure of these series (mainly the functions defining the *terms* of the series) as well as the convergence criteria for them.

Look back to Activity 8.2.1. We noticed that we were able to find an explicit formula for the nth partial sum, which allowed us to evaluate $\lim_{n\to\infty} S_n$. We noticed this again in Example 8.2.3.

But there are some differences between *why* we were able to find formulas for the nth partial sum in each of these examples. Let's first focus on the infinite series with terms defined by exponential functions.

8.6.1 Geometric Series

Definition 8.6.1 Geometric Series. For real numbers a and r with $a, r \neq 0$, we say that the series

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + ...$$

is a **geometric series**. We call r the **constant ratio** and a the **initial term**. ♢

Theorem 8.6.2 Geometric Series Convergence Criteria. *A geometric series $\sum_{k=0}^{\infty} ar^k$ converges to $\frac{a}{1-r}$ when $|r| < 1$ and diverges if $|r| \geq 1$.*

8.6.2 p-Series

Definition 8.6.3 p-Series. For a real number p, we say that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

is a **p-series**. We mostly will be concerned about the case where $p > 0$, making the terms of the series be reciprocal power functions. ♢

Theorem 8.6.4 p-Series Convergence Criteria. *A p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges when $p > 1$ and diverges when $p \leq 1$.*

Proof. Let's divide this into four cases: when $p \leq 0$, when $0 < p < 1$, when $p = 1$, and when $p > 1$.

Case 1: $p \leq 0$

Note that for $\frac{1}{k^p}$ with $p < 0$, we can write this as $k^{|p|}$. Now we can consider the limit of the terms, in order to use the Divergence Test.

$$\lim_{k\to\infty} \frac{1}{k^p} = \lim_{k\to\infty} k^{|p|}$$

Since this limit is non-zero (since it is either ∞ or 1, depending on whether $p = 0$ or not), the series diverges by the Divergence Test.

Case 2: $0 < p < 1$

When $0 < p < 1$, we can apply the Integral Test to the series. It is worth showing that the conditions of the test are met, but this is left up to the reader.
So now we will consider the integral $\int_{x=1}^{\infty} \frac{1}{x^p}\, dx$ as a way of seeing whether the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges or diverges.

$$\begin{aligned}\int_{x=0}^{\infty} \frac{1}{x^p}\, dx &= \lim_{t\to\infty} \int_{x=1}^{x=t} \frac{1}{x^p}\, dx \\ &= \lim_{t\to\infty} \left(\frac{x^{1-p}}{(1-p)}\right)\Bigg|_{x=1}^{x=t} \\ &= \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}\end{aligned}$$

We can note that since $0 < p < 1$, that $1 - p > 0$. This means that when $t \to \infty$, $t^{1-p} \to \infty$ as well.

$$\int_{x=0}^{\infty} \frac{1}{x^p}\, dx \ = \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p} = \infty$$

This integral diverges, and so then does the series.
Case 3: $p = 1$
This is the Harmonic Series! This series diverges (Theorem 8.3.2).
Case 4: $p > 1$
We can repeat the proof from *Case 2*, but we will end with a different conclusion based on the sign of the exponent! Let us, again, apply the Integral Test.
Consider the integral $\int_{x=1}^{\infty} \frac{1}{x^p}\, dx$ as a way of seeing whether the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges or diverges.

$$\begin{aligned}\int_{x=0}^{\infty} \frac{1}{x^p}\, dx &= \lim_{t\to\infty} \int_{x=1}^{x=t} \frac{1}{x^p}\, dx \\ &= \lim_{t\to\infty} \left(\frac{x^{1-p}}{(1-p)}\right)\Bigg|_{x=1}^{x=t} \\ &= \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}\end{aligned}$$

Now, though, we have $p > 1$ which means that $1 - p < 0$. This means that $t^{1-p} = \frac{1}{t^{|p-1|}}$. So now we will consider the limit, and note that as $t \to \infty$, we get $\frac{1}{t^{|p-1|}} \to 0$.

$$\int_{x=0}^{\infty} \frac{1}{x^p}\, dx \ = \lim_{t\to\infty} \frac{1}{(1-p)t^{|p-1|}} - \frac{1}{1-p} = -\frac{1}{1-p}$$

This integral converges, and so then does the series. We remember, though, that the series converges to something different than the integral, and so we do not know what the series converges to. ■

8.6.3 Recapping Our Mathematical Objects

Activity 8.6.1 (Im)Possible Combinations. When we have thought about infinite series, we have thought about three different mathematical objects: the sequence of terms of the series, the sequence of partial sums of the series, and the infinite series itself. As a reminder, if we had an infinite series

$$\sum_{k=1}^{\infty} a_k$$

we can say that:

- $\{a_k\}_{k=1}^{\infty}$ is the sequence of terms of the series
- $S_n = \sum_{k=1}^{n} a_k$ is a partial sum and $\{S_n\}_{n=1}^{\infty}$ is the sequence of partial sums of the series

For each of these three objects -- the terms, the partial sums, and the series -- we have some notion of what it means for that object to converge or diverge.

Consider the following table of all of the different combinations of convergence and divergence of the three objects. For each combination, decide whether this combination is possible or impossible. If it is possible, give an example of an infinite series whose terms, partial sums, and the series itself converge/diverge appropriately. If it is impossible, give an explanation of why.

Table 8.6.5 (Im)Possible Combinations

$\{a_k\}_{k=1}^{\infty}$	$\{S_n\}_{n=1}^{\infty}$	$\sum_{k=1}^{\infty} a_k$	**(Im)Possible?**	**Example or Explanation**
Converges	Converges	Converges		
Converges	Converges	Diverges		
Converges	Diverges	Converges		
Converges	Diverges	Diverges		
Diverges	Converges	Converges		
Diverges	Converges	Diverges		
Diverges	Diverges	Converges		
Diverges	Diverges	Diverges		

1. We can think back to some results or definitions that connect pairs of these objects. Can you think of any result or definition that connects an infinite series and a sequence of partial sums? What about a result or definition that connects the sequence of terms with the infinite series?

2. Look back at Definition 8.2.2 and Theorem 8.3.3.

Table 8.6.6 (Im)Possible Combinations

(Im)Possible?	**Example or Explanation**
Possible	Any converging series serves as an example.
Impossible	The sequence of partial sums and the infinite series are the same object, and so must beh
Impossible	The sequence of partial sums and the infinite series are the same object, and so must beh
Possible	
Impossible	If the infinite series converges, then the sequence of terms must converge to 0. Theorem 8
Impossible	The sequence of partial sums and the infinite series are the same object, and so must beh
Impossible	Both of the reasons, Definition 8.2.2 and Theorem 8.3.3 apply here!
Possible	

8.7 Comparison Tests

So far, our strategies for thinking about infinite series have been focused around drawing a connection between the infinite series we care about and some other mathematical object:

- The Divergence Test draws a connection (even thought it's a limited one) between the terms of the series and the series itself.
- The Alternating Series Test draws a (stronger) connection between the terms of, specifically, an Alternating Series and the series itself.
- The Integral Test draws a connection between the series and a corresponding integral.

Now we'll work on building the most important series convergence test mechanism: we'll draw a link between the series we care about and some other series that we already know about.

This is helpful for three reasons:

1. We already have a couple of types of series that we know about (Section 8.6), and we can keep adding to that list.
2. We can take advantage of similar structure or common term formulas when we see them to essentially say, "This series kind of looks like one that I recognize. I wonder if they act the same?"
3. We don't always have to integrate things using the Integral Test! Integrating can be hard!

8.7.1 Comparing Partial Sums

We're going to start by trying to do the same thing we did when we build the Integral Test: show that the partial sums are monotonic and bounded and then make use of the Monotone Convergence Theorem.

Theorem 8.7.1 Direct Comparison Test. *If* $\sum_{k=0}^{\infty} a_k$ *and* $\sum_{k=10}^{\infty}$ *are infinite series with positive terms (*$a_k > 0$ *and* $b_k > 0$ *for* $k \geq 0$*) with the ordering* $a_k \leq b_k$ *for* $k \geq 0$*, then:*

- *If* $\sum_{k=0}^{\infty} a_k$ *diverges, then* $\sum_{k=0}^{\infty} b_k$ *also diverges.*
- *If* $\sum_{k=0}^{\infty} b_k$ *converges, then* $\sum_{k=0}^{\infty} a_k$ *also converges.*

8.7.2 Failed Comparisons

8.7.3 Limit Comparison

Theorem 8.7.2 Limit Comparison Test. *If* $\sum_{k=0}^{\infty} a_k$ *and* $\sum_{k=0}^{\infty} b_k$ *are infinite series with positive terms (*$a_k > 0$ *and* $b_k > 0$ *for* $k \geq 0$*), then we can consider* $\lim_{k\to\infty} \frac{a_k}{b_k}$.

- *If* $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$*, then:*
  - *If* $\sum_{k=0}^{\infty} a_k$ *diverges, then* $\sum_{k=0}^{\infty} b_k$ *diverges as well.*
  - *If* $\sum_{k=0}^{\infty} b_k$ *converges, then* $\sum_{k=0}^{\infty} a_k$ *converges as well.*
- *If* $\lim_{k\to\infty} \frac{a_k}{b_k} = \infty$*, then:*
  - *If* $\sum_{k=0}^{\infty} a_k$ *converges, then* $\sum_{k=0}^{\infty} b_k$ *converges as well.*
  - *If* $\sum_{k=0}^{\infty} b_k$ *diverges, then* $\sum_{k=0}^{\infty} a_k$ *diverges as well.*

- *If* $\lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = L$ *where* L *is some non-zero real number, then* $\sum_{k=0}^{\infty} a_k$ *and* $\sum_{k=0}^{\infty} b_k$ *will either both converge or both diverge.*

Theorem 8.7.3 *If* a_k *is a rational function of* k*,* $a_k = \frac{p(k)}{q(k)}$ \where *both* $p(k)$ *and* $q(k)$ *are polynomial functions, then:*

- *If* $\deg(q(k) - p(k)) > 1$*, then* $\sum_{k=1}^{\infty} a_k$ *converges.*
- *If* $\deg(q(k) - p(k)) \leq 1$*, then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

8.8 The Ratio and Root Tests

Introduction, remind that the Limit Comparison Test is super useful for rational functions and things that act like p-Series.

8.8.1 Eventually Geometric-ish

We'll look at series that act like geometric series, but first let's remind us what Geometric Series act like.

Activity 8.8.1 Reminder about Geometric Series. We are going to build some convergence tests that try to link some infinite series to the family of geometric series and show that even though a series is *not* geometric, it might act enough like one to be considered "eventually geometric-ish."

But first, what does it mean for a series to be a geometric series?

(a) Describe a defining characteristic of a geometric series. What makes it geometric?

(b) Can you describe this characteristic in another way? For instance, if you described a geometric series using a characteristic about the Explicit Formula, can you describe the same characteristic in the context of the Recursion Relation instead? Or vice versa?

Hint 1. What kinds of functions do we see in the formula for the terms of a geometric series?

Hint 2. How do you describe how you might get from one term in a geometric series to the next one?

(c) Write out a generalized and simplified form of the term a_k of a geometric series explicitly and recursively. In each case, solve for r, the ratio between terms.

Theorem 8.8.1 Root Test. *Let* $\sum_{k=0}^{\infty} a_k$ *be an infinite series with* $a_k > 0$ *for* $k \geq 0$ *and consider* $\lim_{k\to\infty} \sqrt[k]{a_k}$*.*

- *If there is some real number* r *with* $r = \lim_{k\to\infty} \sqrt[k]{a_k}$ *and* $0 \leq r < 1$*, then the series* $\sum_{k=0}^{\infty} a_k$ *converges.*
- *If there is some real number* r *with* $r = \lim_{k\to\infty} \sqrt[k]{a_k}$ *and* $r >$ *or if* $\lim_{k\to\infty} \sqrt[k]{a_k}$ *does not exist, then the series* $\sum_{k=0}^{\infty} a_k$ *diverges.*
- *If* $\lim_{k\to\infty} \sqrt[k]{a_k} = 1$ *then the Root Test fails and is inconclusive.*

Theorem 8.8.2 Ratio Test. *Let* $\sum_{k=0}^{\infty} a_k$ *be an infinite series with* $a_k > 0$ *for* $k \geq 0$ *and consider* $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$*.*

- *If there is some real number* r *with* $r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ *and* $0 \leq r < 1$, *then the series* $\sum_{k=0}^{\infty} a_k$ *converges.*
- *If there is some real number* r *with* $r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ *and* $r >$ *or if* $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ *does not exist, then the series* $\sum_{k=0}^{\infty} a_k$ *diverges.*
- *If* $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$ *then the Root Test fails and is inconclusive.*

Chapter 9

Power Series

9.1 Polynomial Approximations of Functions

9.2 Taylor Series

Text of section.

9.3 Properties of Power Series

Text of section.

9.4 How to Build Taylor Series

Text of section.

9.5 How to Use Taylor Series

Text of section.

Appendix A

More on Limits

Text here.

Colophon

This book was authored in PreTeXt.